

# APPLICATIONS OF REALIZATIONS (AKA LINEARIZATIONS) TO FREE PROBABILITY

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## Abstract.

A powerful technique in free probability called “linearization” allows one to compute probability distributions of functions  $f$  of independent random  $N \times N$  matrices as  $N \rightarrow \infty$ . In the free probability context this linearization has been applied successfully to  $f$  which are noncommutative polynomials. “Linearization” it turns out has been highly developed for about 50 years in a variety of areas ranging from system engineering to automata theory to ring theory. Indeed a linearization is often called a noncommutative system realization. These apply successfully to  $f$  which are rational functions in noncommuting variables  $x_1, \dots, x_g$ .

There is an extensive family of results in this context, and one would like to apply these readily to free probability. One difficulty with this is that free probability requires evaluation on operators in an algebra  $\mathcal{A}$  to be well behaved. We shall see (using deep results of P.M. Cohn) that if  $\mathcal{A}$  is a stably finite  $C^*$ -algebra (eg. one with a faithful trace) or a finite von Neumann algebra, then evaluation behaves well, so classical and modern realization results can be applied directly to free probability set in a type  $\text{II}_1$  factor.

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## 1. INTRODUCTION

In free probability theory [VDN92, HP00, NS06] the calculation of the distribution of polynomials  $p(X_1, \dots, X_g)$  of  $g$  free random variables  $X_1, \dots, X_g$  – or alternatively, the calculation of the asymptotic,  $N \rightarrow \infty$ , eigenvalue distribution of polynomials  $p(X_1^{(N)}, \dots, X_g^{(N)})$  in  $g$  independent  $N \times N$  random matrices – has made essential progress recently [BMS13]. This relies crucially on the so-called linearization trick: a polynomial like  $p(X_1, X_2) = X_1X_2 + X_2X_1$  can also be written in the form

$$X_1X_2 + X_2X_1 = -uQ^{-1}v = -\begin{pmatrix} X_1 & X_2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix},$$

resulting in the fact that the matrix of polynomials

$$(1.1) \quad \hat{p} = \begin{pmatrix} 0 & u \\ v & Q \end{pmatrix} = \begin{pmatrix} 0 & X_1 & X_2 \\ X_1 & 0 & -1 \\ X_2 & -1 & 0 \end{pmatrix},$$

contains all relevant information about our polynomial  $p$ . This  $\hat{p}$  is now a matrix-valued polynomial in the variables, but has the decisive advantage that all entries have degree at most 1. This means that we can write  $\hat{p}$  as the sum of a matrix in  $X_1$  and a matrix in  $X_2$ . For dealing with sums of (operator-valued) free variables, however, one has a quite well-developed analytic machinery in free probability theory. The application of this machinery gives then results as in Fig. 1.

The crucial ingredient in the above, which allowed transforming an unmanageable polynomial into something linear, and thus into something in the realm of powerful free probability techniques, was the mentioned linearization trick. This trick is purely algebraic and does not rely on the freeness of the variables involved. In the free probability community this linearization was introduced and used with powerful applications in the work of Haagerup and Thorbjørnsen [HT05, HST06], and then it was refined to a version as needed for the above presented problem by Anderson [And13]. However, as it turned out this trick in its algebraic form is not new at all, but has been highly developed for about 50 years in a variety of areas ranging from system engineering to automata to ring theory; indeed it is often called a “noncommutative system realization”.

The aim of the present note is to bring these different communities together; in particular, to introduce on one side the control community to a powerful application of realizations and to provide on the other side the free probability and operator algebra community with some background on the extensive family of results in this context.

Apart from trying to give a survey on the problems and results from the various communities and show how they are related there are at least two new contributions arising from this endeavor:

- In the realization context one is actually not solely interested in polynomials; the linearization given in the example above, though having quite non-trivial implications in free probability, is a trivial one from

their point of view. The real gist are (non-commutative) rational functions. This makes it clear that the linearization philosophy is not restricted to polynomials but works equally well for rational functions. This results in the possibility to calculate also the distribution of rational functions in free random variables. Such results are presented here for the first time.

- From a non-commutative analysis point of view rational functions are non-commutative functions whose domain of definition are usually matrices of all sizes. However, the natural setting for our non-commutative random variables are operators on an infinite dimensional Hilbert space, but still equipped with a trace. This means we want to plug in as arguments in our rational functions not matrices but elements from stably finite operator algebras (stably finite  $C^*$ -algebras or finite von Neumann algebras). This gives a quite new perspective on non-commutative functions and raises a couple of canonical new questions - some of which we will answer here.

Whereas it is clear what a non-commutative polynomial  $p(x_1, \dots, x_g)$  in  $g$  non-commuting variables  $x_1, \dots, x_g$  is and how one can apply such a polynomial as a function  $p(X_1, \dots, X_g)$  to any tuple  $(X_1, \dots, X_g)$  of elements from any algebra, the corresponding questions for non-commutative rational functions are surprisingly much more subtle. (In the following it shall be understood that we talk about “non-commutative” rational functions, and we will usually drop the adjective.) Intuitively, it is clear that a rational function  $r(x_1, \dots, x_g)$  should be anything we can get from our variables  $x_1, \dots, x_g$  by algebraic manipulations if we also allow taking inverses. Of course, one should not take the inverse of 0. But it might not be obvious if a given expression is zero or not. For example, it is clear that we cannot invert  $1 - x_1 x_1^{-1}$  within rational functions. However, rational expressions can be very complicated (in particular, involve nested inversions) and deciding whether and how it can be reduced to zero is not so obvious. So it is probably not clear to the reader whether

$$r(x_1, x_2, x_3) = x_2^{-1} + x_2^{-1}(x_3^{-1}x_1^{-1} - x_2^{-1})^{-1}x_2^{-1} - (x_2 - x_3x_1)^{-1}$$

is identically zero or not, and thus whether  $r(x_1, x_2, x_3)^{-1}$  is a valid rational function or not. So one of the main problems in defining and dealing with rational functions is to find a way of deciding whether two rational expressions represent the same function; the underlying idea is of course that they are the same if we can transform one into the other by algebraic manipulations. However, this is not very handy for concrete questions and also a bit clumsy for developing the general theory. There exist actually quite a variety of different approaches to deal with this, resulting in different definitions of rational functions. Of course, in the end all approaches are equivalent, but it is sometimes quite tedious to arrive at this conclusion.

In this paper we will restrict to rational functions which are “regular” at one fixed point in  $\mathbb{C}$  (which we will usually choose to be the point 0). These rational functions can be identified with a subclass of power series and the

theory becomes relatively easy and quite straightforward to handle. In this setting the main ideas and results can in principle be traced back to the work of Schützenberger [S61]. In the setting without this restriction the whole theory gets a more abstract algebraic flavour and relies on the basic work of Cohn on the free field [Co71].

Whichever approach one takes, at one point one arrives at the fundamental insight that it is advantageous to represent rational functions in terms of matrices of linear polynomials of our variables. In the approach we take in this paper the possibility of such a representation is a basic theorem – this is the content of the linearization trick in the free probability context and goes under the name of “realization” in the control community. In the more general algebraic approach according to Cohn this matrix realization is more or less the definition of the free field.

In any case, this matrix realization of a rational function  $r$  is of the form

$$r = C\Lambda^{-1}B,$$

where, for some  $n \in \mathbb{N}$ ,  $\Lambda$  is an  $n \times n$  matrix,  $C$  is an  $1 \times n$  matrix and  $B$  is an  $n \times 1$  matrix, all with polynomials as entries. Actually, in  $\Lambda$  those polynomials can always be taken of degree less or equal 1, whereas the entries of  $C$  and  $B$  can be chosen as constants; such a realization is called “linear” (or a linearization). We will only consider linear realizations.

Of course, also in this realization it will be crucial to decide whether we have a meaningful expression or not; i.e., we must be able to decide whether the inverse of our matrix  $\Lambda$  of polynomials makes sense. However, it turns out that this can actually be decided by an algebraic condition on the level of matrices of polynomials alone. Namely a matrix of polynomials is invertible within matrices of rational functions, if and only if the matrix is full. A matrix is not full if it can be decomposed as a product of smaller strictly rectangular matrices; here it is important that in this factorization only polynomial (and not rational) entries are required. This is a non-obvious and maybe surprising fact in the theory, which lies behind some of its nice structure. In any case, the message is that for deciding whether something is a valid expression within rational functions it suffices to answer a question within the ring of matrices over polynomial functions.

The collection of all non-commutative rational functions gives a skew field, which is also called “free field” and usually denoted by  $\mathbb{C}\langle x_1, \dots, x_g \rangle$ . Now we want to treat the elements from this abstract algebraic object as functions, i.e., we want to evaluate them on tuples of operators  $X_1, \dots, X_g$  from some fixed algebra  $\mathcal{A}$ . Of course, the issue of the domain of the function is here the relevant one.

Again, it is intuitively clear what we mean with the domain of a rational function. If we represent our function by a rational expression, then the domain of this should be all tuples of operators from  $\mathcal{A}$  which we can insert in our rational expression such that all operations make sense. In particular, we should not take the inverse of the zero operator. So we are facing the problem that we have to decide whether a given expression applied to some operators

is zero or not. This rises a couple of questions. First of all, we have to check whether being zero in  $\mathbb{C}\langle x_1, \dots, x_g \rangle$  implies also being zero when applied to our operators. That this is not an trivial issue is shown by the following example. In  $\mathbb{C}\langle x_1, x_2 \rangle$  we clearly have

$$x_1(x_2x_1)^{-1}x_2 - 1 = 0.$$

However, if we take operators  $X_1$  and  $X_2$  with

$$X_2X_1 = 1, \quad \text{but} \quad X_1X_2 \neq 1,$$

then we have for those

$$X_1(X_2X_1)^{-1}X_2 - 1 = X_1X_2 - 1 \neq 0.$$

Hence a rational function being zero in the free field does not imply that it evaluates to zero on every algebra. Luckily, it turns out that the above counter example is essentially the only obstruction for this. One of the basic insights of Cohn (see [Co06]) is that if we consider an algebra which is “stably finite” (sometimes also called “weakly finite”) – this is by definition an algebra where in the matrices over the algebra any left-inverse is also a right-inverse – then relations in the free field will also be relations in the algebra (provided they make sense). We will elaborate on this fact, in our setting, in Sect. 2.4. Stably finite is of course a property which resonates well with operator algebraists. Stably finite  $C^*$ -algebras are a prominent class of operator algebras and on the level of von Neumann algebras this corresponds to finite ones, i.e., those where we have a trace. In our free probability context we usually are working in a finite setting, thus this is tailor-made to our purposes. Of course, from an operator algebraic point of view, type III von Neumann algebras or purely infinite  $C^*$ -algebras are also of much interest, but the above shows that taking rational functions of operators in such a setting might not be a good idea.

Let us now fix a stably finite algebra  $\mathcal{A}$ , so that we can be sure that applying a rational function  $r \in \mathbb{C}\langle x_1, \dots, x_g \rangle$  to operators from  $\mathcal{A}$  is well-defined as long as it makes sense. This making sense is now the issue of domains. Of course, for given operators  $(X_1, \dots, X_g)$  not every rational function  $r$  can be applied to those operators. For example,  $r(x_1) = x_1^{-1}$  makes only sense for invertible operators. However, it is also clear that the domain depends on the rational expression we have chosen to represent our rational function. So,  $r_1(x_1) = x_1 + x_1x_1^{-1}$  has again only invertible operators in its domain, but the better representation as  $r_2(x_1) = x_1 + 1$  shows that this restriction was somehow artificially, and is owed more to the bad choice of representation than to a property of our function. Clearly, there should be a maximal domain, as the union of the domains of all rational expressions representing the considered rational functional. It is not off-hand clear, though, whether there exists a rational expression which has this maximal domain.

However, things become cleaner if we realize our rational functions in terms of matrices in the form  $r(x_1, \dots, x_g) = C\Lambda(x_1, \dots, x_g)^{-1}B$ , with  $\Lambda$  being an  $n \times n$  matrix with affine linear polynomials in the variables as entries. Applying this representation to  $X_1, \dots, X_g \in \mathcal{A}$  is now obvious: the tuple  $(X_1, \dots, X_g)$

is in the domain of  $r$  if  $\Lambda(X_1, \dots, X_g)$  is invertible in  $M_n(\mathcal{A})$ ; and in this case the value of  $r(X_1, \dots, X_g)$  is of course given by

$$r(X_1, \dots, X_g) = C\Lambda(X_1, \dots, X_g)^{-1}B.$$

This has not only the advantage that we have to invert only once in  $M_n(\mathcal{A})$ , but it turns out that there is also a cleaner way for realizing the maximal domain; namely, there is always a “minimal” linear representation for our rational function of the form  $r = \tilde{C}\tilde{\Lambda}^{-1}\tilde{B}$ , such that  $\tilde{\Lambda}(X_1, \dots, X_g)$  is invertible if and only if  $(X_1, \dots, X_g)$  is in the maximal domain of  $r$ .

Let us clarify those last remarks by the following example. Consider the rational function which is given by the following linear realization

$$r(x_1, x_2, x_3, x_4) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then the domain of this rational function are tuples  $(X_1, X_2, X_3, X_4)$  in  $\mathcal{A}$ , for which the matrix

$$\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$$

is invertible in  $M_2(\mathcal{A})$ ; in which case the value of our rational function for those operators is given by

$$r(X_1, X_2, X_3, X_4) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

However, there is no global (scalar as opposed to matrix of) rational expression for capturing the whole domain. The Schur complement formulas give us such expressions, but, according to the chosen pivot, we have different expressions, and those have different domains. So we have for example

$$r_1(x_1, x_2, x_3, x_4) = x_1^{-1} + x_1^{-1}x_2(x_4 - x_3x_1^{-1}x_2)^{-1}x_3x_1^{-1}$$

or

$$r_2(x_1, x_2, x_3, x_4) = (x_1 - x_2x_4^{-1}x_3)^{-1}.$$

Both  $r_1$  and  $r_2$  correspond to the same rational function  $r$ , but they have different domains, and none of them has the maximal domain.

The paper is organized as follows. In Section 2, we first give a brief introduction to the general theory of noncommutative rational functions. In contrast to this classical theory, where noncommutative rational functions are mostly treated as formal objects rather than as actual functions, we will subsequently turn our attention to the seemingly unexplored question of evaluating rational expressions and rational functions on elements coming from general algebras. Thus, as an extension of this well-established theory, we provide here a new framework for treating such questions. As we will see, the crucial condition that guarantees that rational functions behave well under evaluation is that the underlying algebra is stably finite; see Theorem 2.3. Some of these discussions are outsourced to the Appendix 6.

Section 3 describes system realizations for NC multi-variable rational functions, extending the classical work of Schützenberger [S61]. M. Fliess [F74a]



subsequently used Hankel operators effectively in such realizations. See the book [BR84] for a good exposition. A basic reference in the operator theory community is J. Ball, T. Malakorn, and G. Groenewald [BMG05], and more recently D. S. Kaliuzhnyi-Verbovetskyi and V. Vinnikov [KVV12b]. To the well-known theory of descriptor realizations for rational expressions which are regular at 0, i.e., for which their domain contains the point 0, we contribute here with Lemma 3.9 the very important observation that the minimal selfadjoint descriptor realization of a selfadjoint rational expression contains – again under the stably finite hypothesis – the domain of all other selfadjoint descriptor realizations and thus has the largest domain among all of them. This extends the corresponding result of Kaliuzhnyi-Verbovetskyi and Vinnikov on matrix algebras [KVV09].

In particular, as we will see in Theorem 5.15, Lemma 3.9 allows us to conclude that, over any stably finite algebra  $\mathcal{A}$ , the  $\mathcal{A}$ -domain of any rational expression  $r$  is contained in the  $\mathcal{A}$ -domain of each minimal realization  $\mathfrak{r}$  of  $r$ .

The proofs of Theorem 2.3 and of Lemma 3.9 rely crucially on results that will be presented in Section 5. Inspired by constructions of Cohn [Co71, Co06] and Malcolmson [Mal78], we introduce there for general rational expressions (i.e. without the regularity constraint) the notion of formal linear representations. If applied to regular rational expressions, formal linear representation allow us in particular to construct easily descriptor realizations whose domains are larger than the domains of the corresponding rational expressions.

Section 4 is then devoted to several applications of descriptor realizations in the context of free probability. After a brief introduction to scalar- and operator-valued free probability, where we recall in particular the powerful subordination results about the operator-valued free additive convolution that were obtained in [BMS13], we finally present our main result, Theorem 4.7. Roughly speaking, Theorem 4.7 explains how (matrix-valued) Cauchy transforms of selfadjoint (matrices of) rational expressions  $r$ , evaluated at any selfadjoint point  $X = (X_1, \dots, X_g)$  in the  $\mathcal{A}$ -domain of  $r$  for a  $C^*$ -probability space  $(\mathcal{A}, \phi)$ , can be obtained from matrix-valued Cauchy transforms of affine linear pencils of the form

$$\hat{\Lambda}(X) = \hat{\Lambda}_0 + \hat{\Lambda}_1 X_1 + \dots + \hat{\Lambda}_g X_g,$$

where  $\hat{\Lambda}_0, \hat{\Lambda}_1, \dots, \hat{\Lambda}_g$  are certain selfadjoint matrices over  $\mathbb{C}$ , depending only on  $r$  and not on  $X$ .

In the spirit of [BMS13], where the so-called linearization trick in its selfadjoint version due to Anderson [And13] was used in order to compute the distribution of selfadjoint non-commutative polynomials in freely independent variables, we finally explain how Theorem 4.7 and hence realizations instead of linearizations can be used in order to obtain a complete solution of the two Problems 4.8 and 4.9, asking for an algorithm that allows to compute (at least numerically) distributions and even more Brown measures of rational expressions, evaluated in freely independent selfadjoint elements with given distributions. We conclude with several concrete examples, which are discussed in Subsection 4.7.

## 2. AN INTRODUCTION TO NC RATIONAL FUNCTIONS

At first glance this notation section may look formidable to many readers. We offer the reassurance that much of it lays out formal properties of noncommutative rational functions which merely capture manipulations with functions on matrices which are very familiar to matrix theorists and operator theorists. People in these areas might well want to skip these fairly intuitive basics, on first reading, to move quickly to Section 3 and beyond. Beware, rather unintuitive is Section 2.4, which tells us that rational expressions have good properties when evaluated on a Type  $II_1$  factor.

**2.1. NC Linear Pencils.** Throughout this paper  $x = (x_1, \dots, x_g)$  denotes  $g$  noncommutative indeterminates. Given a matrix  $W$  with entries  $W_{ij}$  and a variable  $x_\ell$ , let  $Wx_\ell = x_\ell W$  denote the matrix with entries given by

$$(Wx_\ell)_{ij} = W_{ij}x_\ell.$$

Given  $m \times d$  matrices  $M_1, \dots, M_g$ , define  $L_M$  by

$$L_M(x) := M_1x_1 + \dots + M_gx_g.$$

A  $m \times d$  **NC linear pencil** (in  $g$  indeterminates) is an expression of the form

$$\Lambda_M(x) := M_0 - L_M(x)$$

where  $L_M(x) = M_1x_1 + \dots + M_gx_g$  and  $M_0, M_1, \dots, M_g$  are  $m \times d$  matrices. As an example, for

$$M_0 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad M_1 := -\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \quad M_2 := -\begin{pmatrix} 5 & 4 \\ 4 & 2 \end{pmatrix},$$

the pencil is

$$\Lambda_M(x) = \begin{pmatrix} 1 + 3x_1 + 5x_2 & 2x_1 + 4x_2 \\ 2x_1 + 4x_2 & -1 + x_1 + 2x_2 \end{pmatrix}.$$

Sometimes we refer to the pencil as  $M_0, M_1, \dots, M_g$ . Frequently we multiply all of the matrices  $M_j$  by a single matrix  $G$  or  $W$ , namely,

$$GL_M(x)W = L_{GMW}(x)$$

where  $GMW := (GM_1W, \dots, GM_gW)$ .

Note the common term linear pencil is a misnomer in that *linear pencils are actually affine linear*, that is, the pencil  $\Lambda_M(x)$  is linear if and only if  $M_0 = 0 = M(0)$ .

**2.1.1. Evaluation of pencils.** The variables  $x_i$  will often be evaluated on  $X_i$  which are square matrices or elements of a particular algebra  $\mathcal{A}$ . For pencils the evaluation rule is

$$(2.1) \quad \Lambda_M(X) = M_0 \otimes I_{\mathcal{A}} - M_1 \otimes X_1 - \dots - M_g \otimes X_g$$

where  $X = (X_1, X_2, \dots, X_g)$  is in  $(\mathcal{A})^g$ . Later evaluation will be discussed in more generality.



**2.2. NC Polynomials and NC Rational Expressions.** NC rational functions suited to our purposes here are described in detail in [HVM06], Section 2 and Appendix A. Our discussion here draws heavily from that.

That process has a certain unavoidable heft to it, and we hope to make this paper accessible to people in operator theory where NC rational functions are manipulated successfully without too much formalism. Thus we give here a brief version of our formalism.

**2.2.1. A Few Words about Words.**  $\mathcal{W}_g$  denotes the free semi-group on the  $g$  symbols  $\{\chi_1, \dots, \chi_g\}$ . As always, we let  $x_1, \dots, x_g$  be  $g$  noncommuting formal variables, and for a word  $w = \chi_{i_1} \dots \chi_{i_k} \in \mathcal{W}_g$  we define  $x^w = x_{i_1} \dots x_{i_k}$ .

Occasionally we consider variables which are formal transposes  $x_j^T$  of a variable  $x_j$ , and words in all of these variables  $x_1, \dots, x_g, x_1^T, \dots, x_g^T$ , often called the words in  $x, x^T$ . If  $w$  is in  $\mathcal{W}_g$ , then  $w^T$  denotes the transpose of a word  $w$ . For example, given the word (in the  $x_j$ 's)  $x^w = x_{j_1} x_{j_2} \dots x_{j_n}$ , the involution applied to  $x^w$  is  $(x^w)^T = x_{j_n}^T \dots x_{j_2}^T x_{j_1}^T$ , which, if the variables  $x_k$  are symmetric, is  $x^{(w^T)} = x_{j_n} \dots x_{j_2} x_{j_1}$ .

**2.2.2. The Ring of NC Polynomials.**  $\mathbb{R}\langle x_1, \dots, x_g \rangle :=$  the ring of noncommutative polynomials over  $\mathbb{R}$  in the noncommuting variables  $x_1, \dots, x_g$ . We often abbreviate  $\mathbb{R}\langle x_1, \dots, x_g \rangle$  by  $\mathbb{R}\langle x \rangle$ . When the variables  $x_k$  are symmetric the algebra  $\mathbb{R}\langle x \rangle$  maps to itself under the involution  $^T$ . For non-symmetric variables the algebra of polynomials in them is denoted

$$\mathbb{R}\langle x_1, \dots, x_g, x_1^T, \dots, x_g^T \rangle \quad \text{or} \quad \mathbb{R}\langle x, x^T \rangle.$$

**2.2.3. Polynomial Evaluations.** If  $p$  is an NC polynomial in the symmetric variables  $x_1, \dots, x_g$  and  $X = (X_1, X_2, \dots, X_g)$  is in  $\mathcal{A}^g$ , the evaluation  $p(X)$  is defined by simply replacing  $x_j$  by  $X_j$ . Also  $p(0_{\mathcal{A}}, \dots, 0_{\mathcal{A}}) = I_{\mathcal{A}} \otimes p(0_{\mathbb{R}}, \dots, 0_{\mathbb{R}}) = I_{\mathcal{A}} \otimes p_0$ , where we typically suppress the subscripts on 0.

**2.2.4. NC Rational Expressions.** We shall ultimately define the notion of a NC rational function regular at 0 in terms of rational expressions. Later in Remark 2.6 we address briefly the issue of removing the regular at 0 restriction.

We use a recursion to define the notion of a **NC rational expression**  $r$  **regular at 0** and its value  $r(0)$  at 0. (For details of this definition see the excellent survey [KVV12].) This class includes polynomials and  $p(0)$  is the value of  $p$  at 0 as in the previous subsection. If  $p(0)$  is invertible, then  $p$  is invertible, this inverse is a NC rational expression regular at 0, and  $p^{-1}(0) = p(0)^{-1}$ . Formal sum and products of NC rational expressions regular at 0 and their value at 0 are defined accordingly. Finally, a NC rational expression  $r$  regular at 0 can be inverted provided  $r(0) \neq 0$ ; this inverse is an NC rational expression, and  $r^{-1}(0) = r(0)^{-1}$ . Note that in general  $r(0)$  can itself be zero for the rational expressions we consider. Only the parts of it which must be inverted are required to be different from 0 at 0.

*Example 2.1.* Consider the rational expression

$$(2.2) \quad r = (1 - x_1)^{-1} + (1 - x_1)^{-1} x_2 [(1 - x_1) - x_2(1 - x_1)^{-1} x_2]^{-1} x_2 (1 - x_1)^{-1}$$

Note it is made from inverses of  $(1 - x_1)$  and  $(1 - x_1) - x_2(1 - x_1)^{-1} x_2$  both of which meet our technical invertible at 0 convention.  $\square$

**2.3. Equivalence Classes and NC Rational Functions.** A difficulty is that two different expressions, such as

$$(2.3) \quad r_1 = x_1(1 - x_2 x_1)^{-1} \quad \text{and} \quad r_2 = (1 - x_1 x_2)^{-1} x_1$$

can be converted to each other with algebraic operations. Thus one needs to specify an equivalence relation on rational expressions. The classical notion used in automata and systems theory is uses formal power series; this is summarized later in Section 2.7. Its use to a free analyst is that the classical theorems we need are proved and stated using this type of equivalence. However, what a free analyst often does is substitute matrices or operators in for the variables and so one needs notions of domain of a rational function and a notion of equivalence based on  $r_1$  and  $r_2$  having the same values when evaluated on the same operators. This type of equivalence is developed in [HMV06] for evaluation on matrices and proved to be the same as classical power series equivalence. This alleviates technical headaches. We now define the terms just discussed.

**2.3.1. The Formal Domain of a Rational Expression.** Here we evaluate NC rational expressions on  $g$ -tuples of elements in an algebra  $\mathcal{A}$ . This requires worrying about the domain of an expression and leads to a (very useful) equivalence relation on rational expressions.

$\mathcal{A}$  denotes an algebra over  $\mathbb{C}$  or  $\mathbb{R}$ . The formal domain in  $\mathcal{A}^g$  of a NC rational expression  $r$ , denoted  $\text{dom}_{\mathcal{A}}(r)$ , is defined inductively. If  $p$  is a polynomial, then it is all of  $\mathcal{A}^g$ . If  $r$  is the inverse of the polynomial  $p$ , then the formal domain of  $r$  is  $\{X \in (\mathbb{S}\mathbb{R}^{n \times n})^g : p(X) \text{ is an invertible matrix}\}$ . The formal domain of a general NC rational expression  $r$ , is equal to the intersection of formal domains  $\text{dom}_{\mathcal{A}}(r_j)$  for the rational expressions  $r_j$  and appropriate domains of inverses of the  $r_k$  which appear in the expression  $r$ .

**2.3.2. Evaluation Equivalence of Rational Expressions.** We can use evaluations to define an equivalence on noncommutative rational expressions which we call evaluation equivalence. Two NC rational expressions  $r$  and  $\tilde{r}$  regular at 0 are  **$\mathcal{A}$ -evaluation equivalent** provided

$$r(X) = \tilde{r}(X) \quad \text{for each} \quad X \in \text{dom}_{\mathcal{A}}(r) \cap \text{dom}_{\mathcal{A}}(\tilde{r}).$$

Of course the domains for some algebras  $\mathcal{A}$  might be small in which case this equivalence is weak.

The usual domains considered in the free analysis context consists of matrices of all sizes. We take the equivalence with respect to those as our definition of a **rational function**. This is justified by the fact that one can show that this equivalence is, at least in the case of rational functions which are regular

at zero, the same as power series equivalence; see Section 2.7. Furthermore, it is also the same as the possibility of changing one expression into the other by algebraic operations; for this see [CR99].

An **NC rational function** or simply **rational function** will be defined to be the  $\mathbb{M}(\mathbb{R})$ -equivalent (or  $\mathbb{M}(\mathbb{C})$ -equivalent in the complex case) rational expressions (also called **matrix-equivalent**), where  $\mathbb{M}(\mathbb{R})$  and  $\mathbb{M}(\mathbb{C})$  are the graded algebras of all square, real or complex, matrices.

$$(2.4) \quad \mathbb{M}(\mathbb{R}) := \{\mathbb{R}^{n \times n} : n > 0\} \quad \text{or} \quad \mathbb{M}(\mathbb{C}) := \{\mathbb{C}^{n \times n} : n > 0\}$$

This corresponds to classical engineering type situations. and we typically use German (Fraktur) font to denote NC rational functions.

In Lemma 16.5 of [HMV06] is shown: If  $\mathcal{A} = \mathbb{R}^{n \times n}$ , then  $\text{dom}_{\mathcal{A}}(r)$ , for  $r$  regular at 0, is a non-empty Zariski open subset of  $\mathcal{A}^g$  containing 0.

**2.4. Evaluation on stably finite algebras  $\mathcal{A}$ .** In the free probability context we are not so much interested in plugging in matrices in our rational functions, but we would like to take operators on infinite dimensional Hilbert spaces as arguments.

As we already alluded to in the Introduction, the domain of our rational functions should be stably finite, otherwise there will be inconsistencies. We want to be here a bit more precise on this.

A **stably finite algebra**  $\mathcal{A}$  is one with the following property for each  $n \in \mathbb{N}$ : every  $A \in M_n(\mathcal{A})$  with either a left inverse or a right inverse has an inverse; i.e., if we have  $A, B \in M_n(\mathcal{A})$ , then  $AB = 1$  implies  $BA = 1$ . Sometimes “stably finite” is also addressed as “weakly finite”. These are suitable for free probability, since there we are usually working in a context, where we have a faithful trace.

**Lemma 2.2.** *A unital  $C^*$ -algebra  $\mathcal{A}$  with a faithful trace  $\tau$  is stably finite.*

The lemma is not surprising to those in the area, though we could not find a pinpoint reference in the literature, hence we include its proof (which is short). Dima Shlyakhtenko pointed out to us that this holds not just for a type II factor but also for its affiliated algebra of unbounded operators.

*Proof of Lemma 2.2.* This is a standard fact in operator algebras; see for example [RLL00] where it shows up as an exercise. For the convenience of the reader, let us give a rough outline of proof. First of all, since we can extend the trace  $\tau$  on  $\mathcal{A}$  in the canonical way to a faithful trace  $\text{tr}_n \otimes \tau$  on  $M_n(\mathcal{A})$ , it suffices to prove the required property for  $n = 1$ . (Here  $\text{tr}_n$  denotes the normalized trace on  $n \times n$ -matrices.)

- (1) If  $U \in \mathcal{A}$  is an isometry, i.e.,  $U^*U = 1$ , then it is unitary, i.e., also  $UU^* = 1$ . This follows because we have

$$0 = \tau(1 - U^*U) = \tau(1 - UU^*)$$

and then the faithfulness of  $\tau$  implies  $1 - UU^* = 0$ .

- (2) Consider  $A, B \in B(H)$  with  $AB = 1$  and  $A = A^*$ . Then  $B^*A = 1$ , hence  $B^* = B^*AB = B$ , and thus  $BA = 1$ . Thus  $B$  is invertible.
- (3) Consider now arbitrary  $A, B \in \mathcal{A}$  with  $AB = 1$ . By polar decomposition, we can write  $B = UP$  with  $U$  a partial isometry and  $P \geq 0$ , in particular  $P^* = P$ . Note that a priori  $U$  might not be in  $\mathcal{A}$ , but only in  $B(H)$ . In any case we have then  $AUP = AB = 1$ . So  $P$  has a left-inverse and thus by the previous item also a right inverse and hence is invertible in  $B(H)$ . This inverse, however, must then belong to  $\mathcal{A}$ . Then  $U = BP^{-1}$  belongs also to  $\mathcal{A}$  and must also be an isometry. By the first item, it must then be a unitary. Hence  $B$  is invertible, i.e., we also have  $BA = 1$ .

□

The following theorem states that relations in the free field are valid in any stably finite algebra in our languages of  $\mathcal{A}$ -equivalences.

**Theorem 2.3.**

- (1) If  $\tilde{r}$  and  $r$  are  $\mathbb{M}(\mathbb{R})$ -evaluation equivalent, then  $\tilde{r}$  and  $r$  are  $\mathcal{A}$ -equivalent provided  $\mathcal{A}$  is a stably finite algebra.
- (2) If  $\mathcal{A}$  is not stably finite then there exists rational expressions  $\tilde{r}$  and  $r$ , which are  $\mathbb{M}(\mathbb{R})$ -evaluation equivalent, and  $X \in \mathcal{A}$  with  $X \in \text{dom}_{\mathcal{A}}(r) \cap \text{dom}_{\mathcal{A}}(\tilde{r})$ , but  $r(X) \neq \tilde{r}(X)$ .

This is a special case of Theorem 7.8.3 in the book [Co06]. A warning is that the terminology and cross-referencing there required to make this conversion is extensive. Hence we give a proof in Section 6.

**2.5. Matrix Valued NC Rational Expressions and Functions.** The notion of rational expression is broadened by using matrix constructions. Indeed, this more general notion is often used in this paper.

**2.5.1. Matrix-valued NC Rational Expressions.** Matrix-valued NC rational expressions regular at 0 are defined by analogy to (scalar-valued) rational expressions. A **matrix-valued NC polynomial** is a NC polynomial with matrix coefficients. All matrix-valued NC polynomials are matrix-valued rational expressions. If  $P$  is a square matrix-valued NC polynomial and  $P(0)$  is invertible, then  $P$  has an inverse  $P^{-1}$  whose formal domain is

$$\text{dom}_{\mathcal{A}}(P^{-1}) = \{X \in \mathcal{A}^g : P(X) \text{ is invertible}\}.$$

Matrix-valued NC rational expressions  $R_1$  and  $R_2$  can be added and multiplied whenever their dimensions allow, with the formal domain of the sum and product equal to the intersection of the formal domains. Finally, a square matrix-valued NC rational expression  $R$  has an inverse as long as  $R(0)$  is invertible. (See Appendix A [HMV06] for details.)

Two  $m_1 \times m_2$  matrix-valued NC rational expressions  $R_1$  and  $R_2$  regular at 0 are equivalent provided they take the same values on their common matrix domain. A **matrix-valued NC rational function regular at 0** is an

equivalence class of matrix-valued NC rational expressions. In particular, the definition of **rational expression regular at 0** is now **amended** to mean  $1 \times 1$  matrix-valued rational expressions regular at 0. Notice that the evaluation of a matrix-valued NC rational expression or power series on a  $g$ -tuple of matrices uses tensor substitution of matrices as explained for pencils in Section 3.1.

We shall use the phrase **scalar rational expression regular at 0** if we want to emphasize the absence of matrix constructions. Often when the context makes the usage clear we drop adjectives such as scalar,  $1 \times 1$ , matrix rational, matrix of rational and the like. Indeed, it is shown in Appendix A.4 of [HMV06] that a  $m_1 \times m_2$ -matrix valued noncommutative rational function is in fact the same as a  $m_1 \times m_2$  **matrix of noncommutative rational functions**, and furthermore, any matrix valued noncommutative rational function can be represented by a matrix of scalar rational expressions “near” any point in its domain.

*Example 2.4.* Consider two rational expressions

$$(2.5) \quad m(x) := \begin{pmatrix} 1 - x_1 & -x_2 \\ -x_2 & 1 - x_1 \end{pmatrix}$$

and  $w(x) = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}$  which has entries

$$w_{11} = (1 - x_1)^{-1} + (1 - x_1)^{-1} x_2 ((1 - x_1) - x_2 (1 - x_1)^{-1} x_2)^{-1} x_2 (1 - x_1)^{-1}$$

$$w_{12} = (1 - x_1)^{-1} x_2 ((1 - x_1) - x_2 (1 - x_1)^{-1} x_2)^{-1}$$

$$w_{21} = ((1 - x_1) - x_2 (1 - x_1)^{-1} x_2)^{-1} x_2 (1 - x_1)^{-1}$$

$$w_{22} = ((1 - x_1) - x_2 (1 - x_1)^{-1} x_2)^{-1}$$

If we substitute for  $x$  an operator tuple  $X \in \mathcal{A}^g$  in the  $\mathcal{A}$ -domain of all of these, then it is an easy (and standard) computation to see that  $w(X) = m(X)^{-1}$ . Thus the nc rational function  $w$  is the inverse of  $m$ .

□

**2.5.2. Symmetric Matrix NC Rational Expressions.** A square matrix  $r$  of scalar NC rational expressions or a square matrix-valued NC rational expression is called **symmetric** if  $r(X^T) = r(X)^T$  for each  $X$  in  $\text{dom}_{\mathcal{A}}(r)$ .

Note that if  $r$  is any square matrix of scalar NC rational expressions, there exists another square matrix  $r^T$  of scalar NC rational expressions, which satisfies  $r(X)^T = r^T(X^T)$  for all real matrices  $X$  of arbitrary size in the domain of  $r$ . This matrix-valued NC rational expression  $r^T$  is uniquely determined if we assume that all formal variables  $x_1, \dots, x_n$  are symmetric, i.e.  $x_j^T = x_j$  for  $j = 1, \dots, n$ , and if we impose on the mapping  $r \mapsto r^T$  the additional conditions that it satisfies in general  $(r_1 + r_2)^T = r_1^T + r_2^T$ ,  $(r_1 \cdot r_2)^T = r_2^T \cdot r_1^T$ , and  $(r^{-1})^T = (r^T)^{-1}$ .

The same holds true in the complex case, with the transpose T replaced by the conjugate transpose  $*$ .

In anticipation of the the machinery of formal linear representations that we will present in Section 5, we note that if  $\rho = (u, Q, v)$  is a formal linear representation of a NC rational expression  $r$ , then a formal linear representation of  $r^*$  is given by  $\rho^* = (v^*, Q^*, u^*)$ . In the case of rational expressions, which are regular at 0, we can clearly use their realizations, as they will be introduced in the subsequent Section 3, instead of formal linear representations. Indeed, if  $\mathfrak{r}(x) = D + C(J - L_A(x))^{-1}B$  is any descriptor realization of a NC rational expression  $r$  being regular at 0, then  $\mathfrak{r}^*(x) = D^* + B^*(J - L_{A^*})^{-1}C^*$  yields a descriptor realization of  $r^*$ .

**2.6. Other constructions.** To the ring of polynomials Cohn [Co71][Chapter 7] associates its universal skew field. It is unique but there are several constructions for it. One approach is what we did here; take rational expressions and put an equivalence relation on them. Because of our restrictions at 0 in our definition of rational expression we constructed above only a subset (though very large) of the universal skew field.

Notice that  $\mathbb{M}(\mathbb{R})$ -equivalence can be also defined for noncommutative rational expressions which are not necessarily regular at the origin. For more details on this construction, see Appendix A.6 of [HVM06]. In [KVV12] it is shown that the rational functions constructed like this coincide with Cohn's universal skew field. We will reconsider the construction and realization issues around the universal skew field in a forthcoming work.

**2.7. Series Equivalence and Rational Functions.** We conclude the parts of this paper devoted to background on rational expressions with a brief description of power series equivalence. An example involving expressions which are realizations will be given in Remark 3.2. Here we restrict attention to rational expressions regular at 0. We shall consider formal power series expansions

$$\sum_{w \in \mathcal{W}_g} r_w x^w$$

of NC rational expressions around 0. As an example, consider the operation of inverting a polynomial. If  $p$  is a NC polynomial and  $p(0) \neq 0$ , write  $p = p(0) - q$  where  $q(0) = 0$ , then the inverse  $p^{-1}$  is the series expansion  $r = \frac{1}{p(0)} \sum_k (\frac{q}{p(0)})^k$ . Clearly, taking successive products, sums and inverses allows us to obtain a NC formal power series expansion for any NC rational expression regular at 0.

We say that two NC rational expressions  $r_1$  and  $r_2$  regular at 0 are **power series equivalent** if their series expansion around 0 are the same. For example, series expansion for the functions  $r_1$  and  $r_2$  in (2.3) are

$$(2.6) \quad \sum_{k=0} x_1(x_2x_1)^k \quad \text{and} \quad \sum_{k=0} (x_1x_2)^k x_1.$$

These are the same series, so  $r_1$  and  $r_2$  are power series equivalent. A **noncommutative rational function regular at 0** is an equivalence class  $\mathfrak{r}$  under the power series equivalence relation and the **series expansion for  $\mathfrak{r}$**  is the series expansion of any representative. The set of these equivalence classes is denoted  $\mathbb{R}\langle x \rangle_{\text{Rat}0}$ .



Similar considerations hold for matrices of NC rational expressions. Two  $m_1 \times m_2$  matrix-valued NC rational expressions  $R_1$  and  $R_2$  each have a power series expansion around 0 whose coefficients are  $m_1 \times m_2$  matrices. These coefficients being equal define power series equivalence of  $R_1$  and  $R_2$ , thereby determining equivalence classes which are characterizes **matrix-valued NC rational function regular at 0**.

In [HBMV06] it is proved that power series equivalence and  $\mathbb{M}(\mathbb{R})$  equivalence is the same for matrices of rational expressions regular at 0.

### 3. REALIZATIONS OF $r$

This section begins with a review of the classical theory of descriptor realizations for NC rational functions tailored to future needs. See the book [BR84] for a more complete exposition and the papers [B01] [BMG05] for recent developments. From the existence of descriptor realizations, a natural argument shows that symmetric NC rational functions have symmetric descriptor realizations. The section finishes with uniqueness results for symmetric descriptor realizations.

Beware that in this section **when we refer to a rational expression (function) we always mean a rational expression (function) regular at 0**.

**3.1. Descriptor Realizations.** Define a **descriptor realization** of a  $d_1 \times d_2$  matrix NC rational function  $\mathfrak{r}$  to be a rational expression

$$(3.1) \quad \mathfrak{r}(x) = D + C(J - A_1x_1 - \cdots - A_gx_g)^{-1}B$$

for  $\mathfrak{r}$ , where  $A_j \in \mathbb{R}^{d \times d}$  for  $j = 1, \dots, g$ ,  $D \in \mathbb{R}^{d_1 \times d_2}$ ,  $C \in \mathbb{R}^{d_1 \times d}$  and  $B \in \mathbb{R}^{d \times d_2}$ . Here  $J$  denotes a  $d \times d$  signature matrix, namely,  $J = J^T$  and  $J^2 = I$ . We emphasize that at this point the  $A_j$  are not required to be symmetric.

A **symmetric descriptor realization** is a descriptor realization with

$$D = D^T, \quad B = C^T, \quad J \text{ and the } A_j \text{ are symmetric matrices.}$$

Clearly, the rational function  $\mathfrak{r}$  corresponding to a symmetric descriptor realization is a symmetric rational function.

A descriptor realization is called **monic** provided  $J = I$ .

Of course one can write (3.1) in **monic form**, namely

$$(3.2) \quad \mathfrak{r}(x) = D + C(I - JA_1x_1 - \cdots - JA_gx_g)^{-1}JB$$

and this can be expressed as the Schur complement of the affine linear expression

$$(3.3) \quad \text{Sys}(JA, JB, C, D)(x) := \begin{pmatrix} D & C \\ JB & -(I - JA_1x_1 - \cdots - JA_gx_g) \end{pmatrix}$$



### 3.1.1. Examples and Evaluations.

*Example 3.1.* Here is an example of a  $1 \times 1$  rational expression in two variables obtained as a descriptor realization.

$$\begin{aligned} \mathfrak{r}(x) &= \begin{pmatrix} 1 & 0 \end{pmatrix} \left( I - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x_1 - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x_2 \right)^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 - x_1 & -x_2 \\ -x_2 & 1 - x_1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned}$$

From Example 2.4 we see that an NC symmetric rational expression representing the same NC rational function as  $\mathfrak{r}$  is

$$w_{11} = (1 - x_1)^{-1} + (1 - x_1)^{-1} x_2 \left( (1 - x_1) - x_2 (1 - x_1)^{-1} x_2 \right)^{-1} x_2 (1 - x_1)^{-1}.$$

□

The tensor product notation (already used in  $L_A(X)$ ) provides a convenient way of expressing the evaluation

$$(3.4) \quad \mathfrak{r}(X) = D \otimes I_n + (C \otimes I_n)[J \otimes I_n - L_A(X)]^{-1}(B \otimes I_n).$$

at  $X \in (\mathbb{SR}^{n \times n})^g$ . Here  $I$  denotes the  $n \times n$  identity where  $n$  is chosen to match the size of  $X$ . We often abbreviate  $B \otimes I_n$  to  $B$  and  $C \otimes I_n$  to  $C$ , although this is an abuse of notation.

*Remark 3.2.* In view of Section 2.7, computing the formal power series expansion, and thus the equivalence class (rational function) to which a given descriptor realization belongs, is straightforward. Indeed,

$$\begin{aligned} \mathfrak{r}(x) &= B^T (I - JL_A(x))^{-1} JB \sim \sum_{n \geq 0} B^T (JL_A(x))^n JB \\ &= B^T JB + \sum_{j=1}^g B^T JA_j JB x_j + \dots \end{aligned}$$

This uses  $A_j x_j B = A_j B x_j$ .

□

*Example 3.3.* We return to the rational expression in Example 3.1.

Note it is straightforward to compute the power series expansion. Also the formal domain of the rational expression  $\mathfrak{r}$  is, by definition exactly those  $X = (X_1, X_2) \in (\mathbb{SR}^{n \times n})^2$  for which

$$\begin{pmatrix} I - X_1 & -X_2 \\ -X_2 & I - X_1 \end{pmatrix}$$

is invertible, and for such  $X$

$$\mathfrak{r}(X) = \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} I - X_1 & -X_2 \\ -X_2 & I - X_1 \end{pmatrix}^{-1} \begin{pmatrix} I \\ 0 \end{pmatrix}.$$

□

3.1.2. *Minimality.* A descriptor realization is **controllable** if the **controllable space** defined by

$$\mathcal{S} := \text{span} \{ \text{Range} (JA)^w JB : \text{all words } w \in \mathcal{W}_g \}$$

is all of  $\mathbb{R}^d$ . It is **observable** provided the **unobservable space**

$$\mathcal{Q} = \{ v \in \mathbb{R}^d : C(JA)^w v = 0 \text{ for all words } w \in \mathcal{W}_g \}$$

is 0. An important property is that both spaces are invariant under  $JA_i$  for each  $i$ . Observability can be expressed as controllability for the transpose system, since  $\mathcal{Q}^\perp = \text{range of } \{ ((JA)^w)^T C^T \}$ . Likewise, controllability is the same as observability for the transpose system. We say that the descriptor realization is **minimal** if it is both observable and controllable. We emphasize that since the system has finite dimensional “statespace”  $\mathbb{R}^d$ , only finitely many words  $w \in \mathcal{W}_g$  are needed in the formulas to produce  $\mathcal{S}$  and  $\mathcal{Q}$ .

Two minimal monic descriptor realizations with the same feed through term  $D$ ,

$$\begin{aligned} \mathfrak{r} &= D + C(I - L_A(x))^{-1}B \\ \tilde{\mathfrak{r}} &= D + \tilde{C}(I - \tilde{L}_{\tilde{A}}(x))^{-1}\tilde{B}, \end{aligned}$$

for the same rational function are **similar** provided there exists an invertible matrix  $S$  such that

$$(3.5) \quad SA_j = \tilde{A}_j S, \quad SB = \tilde{B}, \quad C = \tilde{C}S.$$

The  $S$  is known as a **similarity transform**.

**3.2. Properties of Descriptor Realizations.** That noncommutative rational functions regular at 0 have descriptor realizations can be found in [BR84]. Moreover, any two minimal descriptor realizations with the same feed through term  $D$  are similar. The next lemma exploits the symmetry implicit in a symmetric rational function to show, by appropriate choice of similarity transform, that any symmetric noncommutative rational function  $\mathfrak{r}$  regular at 0 has a symmetric minimal descriptor realization; i.e., a symmetric descriptor realization which is minimal amongst all descriptor realizations.

**Lemma 3.4** (Lemma 4.1 [HMOV06]).

- (1) *Any descriptor realization is (more precisely, determines) an NC matrix valued rational function which is regular at 0. Conversely, each  $d_1 \times d_2$  matrix valued NC rational function  $\mathfrak{r}$  regular at 0, has a minimal descriptor realization (which could be taken to be monic) with 0 feed through term ( $D = 0$ ).*  
*Moreover, any two minimal descriptor realizations for  $\mathfrak{r}$  with the same feed through term are similar via a unique similarity transform.*
- (2) *Any NC matrix valued rational function regular at 0 with a symmetric descriptor realization is a symmetric rational function.*
- (3) *If  $\mathfrak{r}$  is a symmetric matrix valued NC rational function regular at 0, then it has a minimal symmetric descriptor realization.*

3.2.1. *Cutting down to get a minimal system.* A construction from classical one variable system theory, which dates back at least to Kalman [K63], also works well in this much more general context, cf. [CR99], [BMG05]. It is that of cutting down the descriptor realization of a rational expression  $r$  to controllability and observability spaces thereby obtaining a minimal realization:

By the cutting down to the controllability space  $\mathcal{S}$  we get a new realization  $\hat{A}, JB, \hat{C}, D$  whose state space is  $\mathcal{S}$  with

$$\hat{A}_i = (JA_i)|_{\mathcal{S}} \quad \hat{C} := C|_{\mathcal{S}}$$

Thus the system  $JA, JB, C, D$  has the following block decomposition wrt. the subspace decomposition  $\mathbb{R}^d = \mathcal{S} + \mathcal{S}^\perp$

$$JA = \begin{pmatrix} \hat{A} & A_{12} \\ 0 & A_{22} \end{pmatrix} \quad C = (\hat{C} \quad C_2)$$

While the system  $\hat{A}, JB, \hat{C}, D$  represents the same rational function as the original system, it may not be observable. However we can repeat the dual of this construction on  $\hat{A}, JB, \hat{C}, D$  and decompose  $\mathcal{S} = \hat{\mathcal{Q}} + \hat{\mathcal{Q}}^\perp$ . This results in a minimal monic descriptor system  $\check{A}, \check{B}, \check{C}, D$  which represents the same rational function (not necessarily the same rational expression) as  $JA$ , a block decomposition has the form given in the next lemma.

**Lemma 3.5.** *With respect to the subspace decomposition  $\hat{\mathcal{Q}} + \hat{\mathcal{Q}}^\perp + \mathcal{S}^\perp$  of  $\mathbb{R}^d$ , the monic system  $JA, JB, C, D$  has the block decomposition.*

$$(3.6) \quad JA = \begin{pmatrix} \hat{A}_{11} & \hat{A}_{12} & A_{12}^1 \\ 0 & \hat{A} & A_{12}^2 \\ 0 & 0 & A_{22} \end{pmatrix} \quad JB = \begin{pmatrix} \hat{B}_1 \\ \check{B} \\ 0 \end{pmatrix} \quad C = (0 \quad \check{C} \quad C_2)$$

Moreover, for any stably finite algebra  $\mathcal{A}$

$$(3.7) \quad \text{dom}_{\mathcal{A}}([I - L_{JA}(x)]^{-1}) \subset \text{dom}_{\mathcal{A}}([I - L_{\check{A}}(x)]^{-1})$$

and the  $d_1 \times d_2$  matrices of rational expressions

$$\mathfrak{r} = D + C(I - L_{JA}(x))^{-1}JB \quad \text{and} \quad \check{\mathfrak{r}} = D + \check{C}(I - L_{\check{A}}(x))^{-1}\check{B},$$

take the same values on any  $X \in \text{dom}_{\mathcal{A}}(\mathfrak{r})$ .

To prove this we need a lemma.

**Lemma 3.6.** *Suppose  $\mathcal{A}$  is a unital  $C^*$ -algebra. Then the statement*

*A  $k \times k$  block triangular matrix  $G$  with entries  $G_{ij}$  in  $M_{m_i \times m_j}(\mathcal{A})$  of compatible sizes is invertible implies the diagonal entries  $G_{ii} \in M_{m_i}(\mathcal{A})$  of  $G$  are invertible*

*holds if and only if  $\mathcal{A}$  is stably finite.*

*Proof.* First prove that stably finite is necessary. Suppose  $R, T \in M_m(\mathcal{A})$  satisfy  $TR = I$  and define

$$(3.8) \quad G = \begin{pmatrix} R & I \\ 0 & T \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} T & -I \\ I - RT & R \end{pmatrix}$$

Then  $G$  and  $W$  are inverses, but if  $\mathcal{A}$  is not stably finite, then for some  $m$  there exist such  $T$  and  $R$  in  $M_m(\mathcal{A})$  which are not invertible.

Next prove stably finite is sufficient. To treat

$$(3.9) \quad G = \begin{pmatrix} G_{11} & * & * & * \\ 0 & G_{22} & * & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & 0 & G_{kk} \end{pmatrix}$$

we shall successively partition  $G$  into 2 blocks which respects the given block structure and also partition the inverse  $W$  of  $G$  conformably with  $G$ :

$$G = \begin{pmatrix} R & S \\ 0 & T \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Calculate that  $GW = I$  implies  $TD = I$ , hence by stable finiteness  $T$  and  $D$  are inverses. That  $WG = I$  implies  $AR = I$ , hence by stable finiteness  $A$  and  $R$  are inverses. Thus both  $R, S$  are invertible and since  $R, T$  are both block triangular we face a similar problem but fortunately with a lower number of blocks. Moreover, it is easy to see that

$$(3.10) \quad G^{-1} = W = \begin{pmatrix} R^{-1} & -R^{-1}ST^{-1} \\ 0 & T^{-1} \end{pmatrix}.$$

Given  $G$  in (3.9) apply this decomposition argument to  $R, T$  and then to subblocks of them, ultimately obtaining the  $G_{ii}$  are invertible.  $\square$

*Proof of Lemma 3.5* The core of the proof of (3.6) was sketched before the lemma statement, but for literature references to the triangular form you can take your choice of contexts, eg. Theorem 1.4 [CR99] or [BMG05] Theorem 8.2. In the simplest of cases the notions of controllability, observability and also the cutdown goes back at least to Rudy Kalman [K63].

Abbreviate  $\Lambda_M(x) = I - L_M(x)$ .

Now to  $\mathcal{A}$  domains. The domain inclusion assertion (3.7) is equivalent to saying

$$(3.11) \quad \Lambda_{JA}(X) = \begin{pmatrix} \Lambda_{\hat{A}_{11}}(X) & \Lambda_{\hat{A}_{12}}(X) & \Lambda_{A_{12}^1}(X) \\ 0 & \Lambda_{\check{A}}(X) & \Lambda_{A_{12}^2}(X) \\ 0 & 0 & \Lambda_{A_{22}}(X) \end{pmatrix}$$

is invertible implies  $\Lambda_{\check{A}}(X)$  must be invertible. Lemma 3.6 analyses this implication and immediately yields (3.7).

That the values of the rational expressions  $\mathfrak{r}$  and  $\check{\mathfrak{r}}$  are equal at  $X \in \mathcal{A}$  for which they are both defined is true because (3.10) applied twice implies

$$\begin{aligned} \mathfrak{r}(X) &= D + C\Lambda_{JA}(X)^{-1}JB \\ &= \begin{pmatrix} 0 & \check{C} & C_2 \end{pmatrix} \begin{pmatrix} \Lambda_{\hat{A}_{11}}(X)^{-1} & * & * \\ 0 & \Lambda_{\check{A}}(X)^{-1} & * \\ 0 & 0 & \Lambda_{A_{22}}(X)^{-1} \end{pmatrix} \begin{pmatrix} \hat{B}_1 \\ \check{B} \\ 0 \end{pmatrix} \end{aligned}$$

which clearly equals  $D + \check{C} \Lambda_{\check{A}}(X)^{-1} \check{B} = \check{\mathfrak{r}}(X)$ .  $\square$

*Remark 3.7.* In preparation for what comes later in Section 6 we record an observation about the special case where a monic system  $JA, JB, C, 0$  is a realization of 0, then  $C\mathcal{S} = 0$  and in terms of cutdowns  $C = \begin{pmatrix} 0 & 0 & C_2 \end{pmatrix}$ . Thus  $Sys(JA, JB, C, 0)$  has the form

$$(3.12) \quad Sys(JA, JB, C, 0) = \tilde{\Pi}_1 \begin{pmatrix} \Lambda_{\hat{A}_{11}} & \Lambda_{\hat{A}_{12}} & B_1 & \Lambda_{A_{12}^1} \\ 0 & \Lambda_{\check{A}} & \check{B} & \Lambda_{A_{12}^2} \\ 0 & 0 & 0 & \Lambda_{A_{22}} \\ 0 & 0 & 0 & \hat{C} \end{pmatrix} \Pi \tilde{\Pi}_2$$

where  $\Pi$  permutes the last two columns and  $\tilde{\Pi}_i$  is the permutation  $\begin{pmatrix} 0 & I_{d_i} \\ I_d & 0 \end{pmatrix}$ .

This is a block  $4 \times 4$  matrix. A block  $n \times n$  matrix which contains a  $\alpha \times \beta$  rectangle of zeroes is called **hollow** (the terminology of P.M. Cohn), if  $\alpha + \beta > n$ . For matrix (3.12) this count is  $6 > 4$ , so it is hollow, a fact which will be used later in Section 6.

**3.2.2. Uniqueness of Symmetric Descriptor Realizations.** There is a useful refinement of the state space similarity theorem for symmetric descriptor realizations.

**Proposition 3.8** (Proposition 4.3 [HMV06]). *If*

$$\mathfrak{r} = D + C(J - L_A(x))^{-1}C^T \quad \text{and} \quad \tilde{\mathfrak{r}} = D + \tilde{C}(\tilde{J} - L_{\tilde{A}}(x))^{-1}\tilde{C}^T$$

*are both minimal symmetric descriptor realizations for the same  $d_1 \times d_2$  matrices of rational functions (with the same symmetric feed through term  $D$ ), then there is an invertible similarity transform  $S$  between the two systems; it satisfies  $S^T \tilde{J} S = J$  and*

$$SJA_j = \tilde{J}\tilde{A}_jS \quad SJC^T = \tilde{J}\tilde{C}^T \quad C = \tilde{C}S.$$

*Thus, if  $J = I$ , then  $\tilde{J} = I$  too and  $S$  is unitary. In particular any two monic ( $J = I$ ) symmetric minimal descriptor realizations with the same feed through term for the same matrix rational function are unitarily equivalent.*

*Proof.* We shall recall the proof from [HMV06], since Proposition 4.3 there was only stated for noncommutative scalar expressions. However, as we now see, the proof works for matrix rational expressions. First put  $\mathfrak{r}$  and  $\tilde{\mathfrak{r}}$  in the form (3.1.2) by multiplying appropriately by  $J$  and  $\tilde{J}$  respectively. Since both  $\mathfrak{r}$  and  $\tilde{\mathfrak{r}}$  represent the same rational function (and share the feed through term  $D$ ). From controllability and observability (from the state space similarity theorem) we know that there is an invertible similarity transform  $S$ ; it satisfies (3.5). Thus

$$SJA_j = \tilde{J}\tilde{A}_jS \quad SJC^T = \tilde{J}\tilde{C}^T \quad C = \tilde{C}S$$

Hence,  $S(JA)^\alpha JC^T = (\tilde{J}\tilde{A})^\alpha \tilde{J}\tilde{C}^T$  and  $C(JA)^\alpha = \tilde{C}(\tilde{J}\tilde{A})^\alpha S$  for all words  $\alpha$ .

Since the  $A_j$  and  $\tilde{A}_j$  are symmetric, it follows that

$$(3.13) \quad CJ(AJ)^{\beta^T} S^T \tilde{J} S (JA)^\alpha JC^T = \tilde{C} \tilde{J} (\tilde{A} \tilde{J})^{\beta^T} \tilde{J} (\tilde{J} \tilde{A})^\alpha \tilde{J} \tilde{C}^T$$

which equals  $\tilde{C}(\tilde{J}\tilde{A})^{\beta^T}(\tilde{J}\tilde{A})^\alpha\tilde{J}\tilde{C}^T$ . The power series equivalence (see Section 2.7) of the rational expressions  $\mathfrak{r}$  and  $\tilde{\mathfrak{r}}$  implies

$$(3.14) \quad C(JA)^w JC^T = \tilde{C}(\tilde{J}\tilde{A})^w \tilde{J}\tilde{C}^T$$

for all words  $w$ . Which we use to obtain  $CJ(AJ)^{(\beta^T)}S^T\tilde{J}S(JA)^\alpha JC^T = C(JA)^{(\beta^T)}(JA)^\alpha JC^T$ . Therefore

$$(3.15) \quad CJ(AJ)^{(\beta^T)}S^T\tilde{J}S(JA)^\alpha JC^T = CJ(AJ)^{(\beta^T)}J(JA)^\alpha JC^T, \text{ so}$$

the controllability and observability implies  $S^T\tilde{J}S = J$ .  $\square$

Beware, if  $J \neq I$  the cutdown system  $\tilde{A}, \tilde{B}, \tilde{C}, D$  in Lemma 3.5 while monic often will not be symmetric. One can, however, symmetrize it as in Lemma 4.2 in Section 4.3 of [HMOV06], notably without changing the size of the matrices  $\tilde{A}, \tilde{B}, \tilde{C}, D$  and even more without changing the feed through term  $D$ . This combines with the above to yield that minimal realizations have the maximal  $\mathcal{A}$ -domains, as we now state formally.

**Lemma 3.9.**

(1) *Any two minimal realizations of the same matrix-valued NC rational expression, which both have the same feed through term, have the same  $\mathcal{A}$ -domain for any unital complex algebra  $\mathcal{A}$ .*

*Furthermore, they take the same values on their joint  $\mathcal{A}$ -domain.*

(2) *Suppose  $J, A, B, C, D$  and  $\hat{J}, \hat{A}, \hat{B}, \hat{C}, D$  are both (symmetric) descriptor realizations for the same (symmetric) matrix-valued NC rational expression with  $\hat{J}, \hat{A}, \hat{B}, \hat{C}, D$  being minimal. If  $\mathcal{A}$  is a unital  $(*)$ -algebra, which is stably finite, then*

$$(3.16) \quad \text{dom}_{\mathcal{A}}([J - L_A(x)]^{-1}) \subset \text{dom}_{\mathcal{A}}([\hat{J} - L_{\hat{A}}(x)]^{-1})$$

*and the matrix NC rational expressions*

$$\mathfrak{r} = D + C(J - L_A(x))^{-1}B \quad \hat{\mathfrak{r}} = D + \hat{C}(\hat{J} - L_{\hat{A}}(x))^{-1}\hat{B},$$

*take the same values on any  $X \in \text{dom}_{\mathcal{A}}(\mathfrak{r})$ .*

*Proof.* The first assertion in Item 1 is an immediate consequence of the state space similarity theorem formulated in Item 1 of Lemma 3.4. Indeed, if

$$\mathfrak{r} = D + C(J - L_A(x))^{-1}C^T \quad \text{and} \quad \tilde{\mathfrak{r}} = D + \tilde{C}(\tilde{J} - L_{\tilde{A}}(x))^{-1}\tilde{C}^T$$

are both minimal descriptor realizations of the same matrix-valued NC rational expression  $r$ , then Item 1 of Lemma 3.4 guarantees that there is an invertible matrix  $S$ , which satisfies

$$SJA_j = \tilde{J}\tilde{A}_jS \quad SJC^T = \tilde{J}\tilde{C}^T \quad C = \tilde{C}S.$$

If we put  $\tilde{S} := \tilde{J}S$ , which gives as well an invertible matrix, then we may check that

$$\tilde{S}(J - L_A(x)) = (\tilde{J} - L_{\tilde{A}}(x))S.$$

Indeed, since  $SJL_A(x) = \tilde{J}L_{\tilde{A}}(x)S$ , we obtain

$$SJ(J - L_A(x)) = S - SJL_A(x) = S - \tilde{J}L_{\tilde{A}}(x)S = \tilde{J}(\tilde{J} - L_{\tilde{A}}(x))S$$

and hence  $\tilde{S}(J - L_A(x)) = (\tilde{J} - L_{\tilde{A}}(x))S$ , as stated. Thus,

$$\text{dom}_{\mathcal{A}}([J - L_A(x)]^{-1}) = \text{dom}_{\mathcal{A}}([\tilde{J} - L_{\tilde{A}}(x)]^{-1}).$$

The proof of Item 2 is based on explicit constructions which can be implemented numerically. We start with  $J, A, B, C, D$  and apply Lemma 3.5 to obtain a monic minimal realization  $I, \tilde{A}, \tilde{B}, \tilde{C}, D$  satisfying domain inclusion as in (3.7), i.e.

$$\text{dom}_{\mathcal{A}}([I - L_{JA}(x)]^{-1}) \subseteq \text{dom}_{\mathcal{A}}([I - L_{\tilde{A}}(x)]^{-1})$$

Furthermore, by Item (1), the cut down system  $I, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$  must have the same  $\mathcal{A}$ -domain as the given minimal system  $\hat{J}, \hat{A}, \hat{B}, \hat{C}, D$ . Thus,

$$\begin{aligned} \text{dom}_{\mathcal{A}}([J - L_A(x)]^{-1}) &= \text{dom}_{\mathcal{A}}([I - L_{JA}(x)]^{-1}) \\ &\subseteq \text{dom}_{\mathcal{A}}([I - L_{\tilde{A}}(x)]^{-1}) = \text{dom}_{\mathcal{A}}([\hat{J} - L_{\hat{A}}(x)]^{-1}). \end{aligned}$$

Moreover, we know

- by Lemma 3.5 that  $\mathfrak{r}$  and  $\check{\mathfrak{r}}$  take the same value on any  $X \in \text{dom}_{\mathcal{A}}(\mathfrak{r})$  and
- by Item (1) that  $\check{\mathfrak{r}}$  and  $\hat{\mathfrak{r}}$  take the same value on any  $X \in \text{dom}_{\mathcal{A}}(\check{\mathfrak{r}})$ ,

thus

$$\mathfrak{r} = D + C(J - L_A(x))^{-1}B \quad \text{and} \quad \hat{\mathfrak{r}} = D + \hat{C}(\hat{J} - L_{\hat{A}}(x))^{-1}\hat{B},$$

take the same values on any  $X \in \text{dom}_{\mathcal{A}}(\mathfrak{r})$ .

We conclude by noting that the symmetric case of Item 2 is clearly covered by the more general statement that was proven above, since any minimal symmetric realization is in particular a minimal realization.  $\square$

#### 4. FREE PROBABILITY

Free probability theory was invented around 1985 by D. Voiculescu as a tool to attack the isomorphism problem for the free group factors  $L(\mathbb{F}_n)$ . Although this initial question is still open, free probability gave deep insights to this problem and provides presently many powerful tools which are also applied in other fields of mathematics like random matrix theory.

Roughly speaking, free probability theory can be seen as a highly non-commutative counterpart of classical probability, where the notion of classical independence is replaced by free independence. In the initial example  $L(\mathbb{F}_n)$ , free independence reflects the structure that is induced on the operator algebraic side by free products on the group side.

The rather surprising connection to random matrix theory is given by Voiculescu's observation, that free independence shows up for many classes of independent random matrices in the limit when their dimension tends to infinity.

For more information on free probability in general we refer the reader to the monographs [VDN92, HP00, NS06].



**4.1. A quick introduction to free probability.** The underlying object of free probability theory are **non-commutative probability spaces**  $(\mathcal{A}, \phi)$ . In the most general case, namely in a purely algebraic setting, it consists of a unital complex algebra  $\mathcal{A}$  and a linear functional  $\phi : \mathcal{A} \rightarrow \mathbb{C}$  that satisfies  $\phi(1) = 1$ . Elements of  $\mathcal{A}$  are called **non-commutative random variables** and we refer to  $\phi$  as **expectation** on  $\mathcal{A}$ .

**Definition 4.1** (Free independence). Let  $(\mathcal{A}, \phi)$  be a non-commutative probability space and let  $(\mathcal{A}_i)_{i \in I}$  be a family of unital subalgebras of  $\mathcal{A}$  with an arbitrary index set  $I \neq \emptyset$ . We call  $(\mathcal{A}_i)_{i \in I}$  **freely independent** (or just **free**), if

$$\phi(X_1 \cdots X_n) = 0$$

holds whenever the following conditions are fulfilled:

- We have  $n \geq 1$  and there are indices  $i_1, \dots, i_n \in I$  satisfying

$$i_1 \neq i_2, \dots, i_{n-1} \neq i_n.$$

- For  $j = 1, \dots, n$ , we have  $X_j \in \mathcal{A}_{i_j}$  and it holds true that  $\phi(X_j) = 0$ .

Elements  $(X_i)_{i \in I}$  are called **freely independent** (or just **free**), if  $(\mathcal{A}_i)_{i \in I}$  are freely independent in the above sense, where  $\mathcal{A}_i$  denotes for each  $i \in I$  the subalgebra of  $\mathcal{A}$  that is generated by 1 and  $X_i$ .  $\square$

Although the powerful combinatorial structure induced by non-crossing partitions, which was revealed by Speicher, can already be exploited in this framework, we will mainly work in the more regular setting of  $C^*$ -probability spaces: if  $\mathcal{A}$  is a  $C^*$ -algebra and  $\phi : \mathcal{A} \rightarrow \mathbb{C}$  a positive state on  $\mathcal{A}$ , we call  $(\mathcal{A}, \phi)$  a **non-commutative  $C^*$ -probability space**.

In this case, there corresponds to any element  $X = X^* \in \mathcal{A}$  a unique probability measure  $\mu_X$  on the real line  $\mathbb{R}$  (compactly supported on the spectrum of  $X$ ), which is determined by the condition that  $\mu_X$  has the same moments as  $X$  with respect to  $\phi$ , i.e.

$$\phi(X^k) = \int_{\mathbb{R}} t^k d\mu_X(t) \quad \text{for all } k \in \mathbb{N}.$$

We call  $\mu_X$  the **distribution** of  $X$ .

It was a fundamental observation of Voiculescu, that the distribution  $\mu_{X+Y}$  of  $X + Y$  for freely independent elements  $X, Y \in \mathcal{A}$  only depends on their distributions  $\mu_X$  and  $\mu_Y$ . Thus, we may write  $\mu_{X+Y} = \mu_X \boxplus \mu_Y$  and we call this operation  $\boxplus$  the **free additive convolution**.

The main tool for calculating the free additive convolution is the **Cauchy transform**, which is in classical probability (up to sign) also known under the name **Stieltjes transform**. The Cauchy transform  $G_\mu$  of any probability measure  $\mu$  on  $\mathbb{R}$  is the regular function defined by

$$G_\mu(z) := \int_{\mathbb{R}} \frac{1}{z - t} d\mu(t) \quad \text{for all } z \in \mathbb{C}^+,$$

where  $\mathbb{C}^+ := \{z \in \mathbb{C} | \Im(z) > 0\}$  denotes the upper half-plane. It is easy to check that  $G_\mu$  maps  $\mathbb{C}^+$  to the lower half-plane  $\mathbb{C}^- := \{z \in \mathbb{C} | \Im(z) < 0\}$ .

Note, that if  $\mu_X$  is the distribution of any non-commutative random variable  $X = X^* \in \mathcal{A}$  in a  $C^*$ -probability space  $(\mathcal{A}, \phi)$ , we have

$$G_{\mu_X}(z) = \int_{\mathbb{R}} \frac{1}{z - t} d\mu_X(t) = \phi((z - X)^{-1}) \quad \text{for } z \in \mathbb{C}^+.$$

Thus, in such cases, we will often write  $G_X$  instead of  $G_{\mu_X}$ .

The **Stieltjes inversion formula** tells us, that  $\mu$  can be recovered from its Cauchy transform  $G_\mu : \mathbb{C}^+ \rightarrow \mathbb{C}^-$ . In fact, the absolutely continuous probability measures  $\mu_\varepsilon$  given by

$$d\mu_\varepsilon(t) = \frac{-1}{\pi} \Im(G_\mu(t + i\varepsilon)) dt$$

converge in distribution to  $\mu$  as  $\varepsilon \searrow 0$ , i.e. we have

$$\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} f(t) d\mu_\varepsilon(t) = \int_{\mathbb{R}} f(t) d\mu(t)$$

for all bounded continuous functions  $f : \mathbb{R} \rightarrow \mathbb{C}$ .

**4.2. Polynomials in free random variables.** The following fundamental question leads naturally to realizations and linearizations, respectively. Given a selfadjoint non-commutative polynomial  $P \in \mathbb{C}\langle x_1, \dots, x_n \rangle$  and freely independent non-commutative random variables  $X_1, \dots, X_n$  in some  $C^*$ -probability space. How can we calculate the distribution of  $P(X_1, \dots, X_n)$  out of the given distributions of  $X_1, \dots, X_n$ ?

Of course, in the case  $n = 2$ , this question for  $P = x_1 + x_2$  is answered by the free additive convolution  $\boxplus$  and the question for  $P = x_1 x_2 x_1$  can be answered by using its multiplicative counterpart  $\boxtimes$ , the **multiplicative free convolution**.

The general case of this question was treated systematically in [BMS13]. Therein, the key idea was to translate the non-linear but scalar-valued problem to a linear but matrix-valued problem, where the latter can be solved by operator-valued free probability theory.

**4.3. Operator-valued free probability and subordination.** The main difference between scalar- and operator-valued free probability theory is that expectations get replaced by conditional expectations. They can be seen as natural non-commutative analogues of conditional expectations known in classical probability.

An **operator-valued  $C^*$ -probability space**  $(\mathcal{A}, E, \mathcal{B})$  consists of a unital  $C^*$ -algebra  $\mathcal{A}$ , a unital  $C^*$ -subalgebra  $\mathcal{B}$  of  $\mathcal{A}$ , and a **conditional expectation**  $E : \mathcal{A} \rightarrow \mathcal{B}$ , i.e. a (completely) positive and unital map  $E : \mathcal{A} \rightarrow \mathcal{B}$  satisfying

- $E[B] = B$  for all  $B \in \mathcal{B}$  and
- $E[B_1 X B_2] = B_1 E[X] B_2$  for all  $X \in \mathcal{A}$ ,  $B_1, B_2 \in \mathcal{B}$ .

The definition of free independence in the operator-valued setting reads as follows.

**Definition 4.2** (Free independence with amalgamation). Let  $(\mathcal{A}, E, \mathcal{B})$  be an operator-valued  $C^*$ -probability space and let  $(\mathcal{A}_i)_{i \in I}$  be a family of subalgebras  $\mathcal{B} \subseteq \mathcal{A}_i \subseteq \mathcal{A}$  with an arbitrary index set  $I \neq \emptyset$ . We call  $(\mathcal{A})_{i \in I}$  **freely independent with amalgamation over  $\mathcal{B}$**  (or just **free over  $\mathcal{B}$** ), if

$$E[X_1 \cdots X_n] = 0$$

holds whenever the following conditions are fulfilled:

- We have  $n \geq 1$  and there are indices  $i_1, \dots, i_n \in I$  satisfying

$$i_1 \neq i_2, \dots, i_{n-1} \neq i_n.$$

- For  $j = 1, \dots, n$ , we have  $X_j \in \mathcal{A}_{i_j}$  and it holds true that  $E[X_j] = 0$ .

Elements  $(X_i)_{i \in I}$  are called **freely independent with amalgamation over  $\mathcal{B}$**  (or just **free with amalgamation over  $\mathcal{B}$** ), if  $(\mathcal{A}_i)_{i \in I}$  are freely independent with amalgamation over  $\mathcal{B}$  in the above sense, where  $\mathcal{A}_i$  denotes for each  $i \in I$  the subalgebra of  $\mathcal{A}$  that is generated by  $\mathcal{B}$  and  $X_i$ .  $\square$

*Remark 4.3.* For our purposes, it is important to note that operator-valued  $C^*$ -probability spaces can easily be constructed by passing to matrices over scalar-valued  $C^*$ -probability spaces. Indeed, if  $(\mathcal{C}, \phi)$  is any non-commutative  $C^*$ -probability space, then

$$\mathcal{A} := M_N(\mathbb{C}) \otimes \mathcal{C}, \quad \mathcal{B} := M_N(\mathbb{C}) \quad \text{and} \quad E := \text{id}_{M_N(\mathbb{C})} \otimes \phi$$

give an operator-valued  $C^*$ -probability space  $(\mathcal{A}, E, \mathcal{B})$ . Moreover, a direct calculation shows that if  $(\mathcal{C}_i)_{i \in I}$  is a family of freely independent subalgebras of  $\mathcal{C}$ , then  $\mathcal{A}_i := M_N(\mathbb{C}) \otimes \mathcal{C}_i$  for  $i \in I$  defines a family  $(\mathcal{A}_i)_{i \in I}$  of subalgebras of  $\mathcal{A}$ , which is freely independent with amalgamation over  $\mathcal{B}$ .  $\square$

We stress that even operator-valued free probability theory possesses like the scalar-valued theory a rich combinatorial structure given by non-crossing partitions, as it was originated by Speicher.

Similar to the scalar-valued case, Cauchy transforms play an important role in the regular theory of free independence with amalgamation, but they need to be generalized in the following way: Let  $(\mathcal{A}, E, \mathcal{B})$  be an operator-valued  $C^*$ -probability space. We call

$$\mathbb{H}^+(\mathcal{B}) := \{B \in \mathcal{B} \mid \exists \varepsilon > 0 : \Im(B) \geq \varepsilon 1\}$$

the upper half-plane of  $\mathcal{B}$ , where we use the notation  $\Im(B) := \frac{1}{2i}(B - B^*)$ . The  **$\mathcal{B}$ -valued Cauchy transform**  $G_X$  of any  $X = X^* \in \mathcal{A}$  is the Fréchet holomorphic function defined by

$$G_X(B) := E[(B - X)^{-1}] \quad \text{for all } B \in \mathbb{H}^+(\mathcal{B}).$$

In fact, it induces a map  $G_X : \mathbb{H}^+(\mathcal{B}) \rightarrow \mathbb{H}^-(\mathcal{B})$  from  $\mathbb{H}^+(\mathcal{B})$  to the lower half-plane  $\mathbb{H}^-(\mathcal{B})$  defined by

$$\mathbb{H}^-(\mathcal{B}) := \{B \in \mathcal{B} \mid \exists \varepsilon > 0 : -\Im(B) \geq \varepsilon 1\}.$$

The most convenient way to deal with free additive convolution, both in the scalar- and in the operator-valued setting, is to use subordination, since it is

easily accessible for numerical computations. Before we give the precise statement, we introduce the following regular transforms, which are both related to Cauchy transforms, namely

- the reciprocal Cauchy transform  $F_X : \mathbb{H}^+(\mathcal{B}) \rightarrow \mathbb{H}^+(\mathcal{B})$  by

$$F_X(B) = E \left[ (B - X)^{-1} \right]^{-1} = G_X(B)^{-1},$$

- and the  $h$  transform  $h_X : \mathbb{H}^+(\mathcal{B}) \rightarrow \overline{\mathbb{H}^+(\mathcal{B})}$  by

$$h_X(B) = E \left[ (B - X)^{-1} \right]^{-1} - B = F_X(B) - B.$$

Note, that these mappings are indeed well-defined since it has been shown in [BPV12] that  $\Im(F_X(B)) \geq \Im(B)$  for all  $B \in \mathbb{H}^+(\mathcal{B})$ , which immediately implies  $\Im(h_X(B)) \geq 0$  for all  $B \in \mathbb{H}^+(\mathcal{B})$ .

**Theorem 4.4** ([BMS13]). *Assume that  $(\mathcal{A}, E, \mathcal{B})$  is a  $C^*$ -operator-valued non-commutative probability space and  $X, Y \in \mathcal{A}$  are two selfadjoint operator-valued random variables free over  $\mathcal{B}$ . Then there exists a unique pair of Fréchet (and thus also Gâteaux) regular maps  $\omega_1, \omega_2 : \mathbb{H}^+(\mathcal{B}) \rightarrow \mathbb{H}^+(\mathcal{B})$  so that*

- (1)  $\Im(\omega_j(B)) \geq \Im(B)$  for all  $B \in \mathbb{H}^+(\mathcal{B})$  and  $j \in \{1, 2\}$ ,
- (2)  $F_X(\omega_1(B)) + B = F_Y(\omega_2(B)) + B = \omega_1(B) + \omega_2(B)$  for all  $B \in \mathbb{H}^+(\mathcal{B})$ ,
- (3)  $G_X(\omega_1(B)) = G_Y(\omega_2(B)) = G_{X+Y}(B)$  for all  $B \in \mathbb{H}^+(\mathcal{B})$ .

Moreover, if  $B \in \mathbb{H}^+(\mathcal{B})$ , then  $\omega_1(B)$  is the unique fixed point of the map

$$f_B : \mathbb{H}^+(\mathcal{B}) \rightarrow \mathbb{H}^+(\mathcal{B}), \quad f_B(W) = h_Y(h_X(W) + B) + B,$$

and  $\omega_1(B) = \lim_{n \rightarrow \infty} f_B^{\circ n}(W)$  for any  $W \in \mathbb{H}^+(\mathcal{B})$ , where  $f_B^{\circ n}$  means the  $n$ -fold composition of  $f_B$  with itself. Same statements hold for  $\omega_2$ , with  $f_B$  replaced by  $W \mapsto h_X(h_Y(W) + B) + B$ .

As we already mentioned in Subsection 4.2, operator-valued free probability provides the main tools for the calculation of the distribution of any selfadjoint polynomial  $p(X_1, \dots, X_n)$  in free selfadjoint variables  $X_1, \dots, X_n$  if the distributions of each of the  $X_j$ 's is given. In fact, the algorithmic solution to this problem, which was given in [BMS13], is based on Theorem 4.4. It is therefore easily accessible for numerical computations.

We will see in the subsequent subsection that for this purpose, Theorem 4.4 has to be combined with certain algebraic tools, which allow to represent the Cauchy transform of any non-commutative polynomial as a matrix-valued Cauchy transform of a linear but matrix-valued polynomial. It is curial that these algebraic methods allow us to find also a selfadjoint representation if the considered non-commutative polynomial is itself selfadjoint.

But before proceeding in this direction, we point out that the same methods, as it was shown in [BSS2013], can also be applied in the case where the polynomial in question is not selfadjoint, namely for calculating the so-called Brown measure of arbitrary non-commutative polynomials in free variables. For reader's convenience, we recall some basic facts about Brown measures in the next subsection.

**4.4. Brown measures.** In the case where the considered polynomial fails to be selfadjoint, we can not ask anymore for its distribution as a probability measure on the real line  $\mathbb{R}$ . Even in the case where the polynomial happens to be at least a normal element, the usual methods break down, since this leads in general to probability measures which are supported on more complicated compact subsets of the complex plane  $\mathbb{C}$ .

However, the so-called Brown measure, which was introduced in [Br86] and revived in [HL00], gives an appropriate replacement in this generality. In order for this theory to work we need to work in the setting of finite von Neumann algebras. The main point is that we need our state  $\phi$  to be a trace. Since this corresponds exactly to the stably finite situation, this goes very well with our observation that we need stably finite algebras as domains of our rational functions. Hence we will in the following in the discussions around the Brown measure always assume that our noncommutative probability space  $(\mathcal{A}, \phi)$  is actually a **tracial  $W^*$ -probability space**, which means that  $\mathcal{A}$  is a von Neumann algebra and  $\phi$  is a faithful, normal trace. Many concrete situations appearing in free probability theory are embedded in a tracial  $W^*$ -probability setting.

Given an arbitrary element  $X$  in any tracial  $W^*$ -probability space  $(\mathcal{A}, \phi)$ , we may define its **Fuglede-Kadison determinant**  $\Delta(X)$  by

$$\Delta(X) := \lim_{\varepsilon \searrow 0} \frac{1}{2} \phi(\log(XX^* + \varepsilon^2)).$$

It was shown in [Br86], that the function  $z \mapsto \frac{1}{2\pi} \log(\Delta(X - z\mathbf{1}))$  is subharmonic on  $\mathbb{C}$  and harmonic outside the spectrum of  $X$ . Thus, the Riesz Decomposition Theorem (cf. [Ran95, Theorem 3.7.9]) shows that there exists a Radon measure  $\mu_X$  on  $\mathbb{C}$  such that

$$\int_{\mathbb{C}} \psi(z) d\mu_X(z) = \frac{1}{2\pi} \int_{\mathbb{C}} \left( \frac{\partial^2 \psi}{\partial x^2}(z) + \frac{\partial^2 \psi}{\partial y^2}(z) \right) \log(\Delta(X - z\mathbf{1})) d\lambda^2(z)$$

holds for all functions  $\psi \in C_c^\infty(\mathbb{C})$ . There, we denote by  $\lambda^2$  the Lebesgue measure on  $\mathbb{C}$ , which is induced under the usual identification of  $\mathbb{C}$  with  $\mathbb{R}^2$ . In other words, the **Brown measure**  $\mu_X$  is the **generalized Laplacian** of  $z \mapsto \frac{1}{2\pi} \log(\Delta(X - z\mathbf{1}))$ , which means that  $\mu_X$  of  $X$  is determined (in the distributional sense) by

$$(4.1) \quad \mu_X = \frac{2}{\pi} \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \log(\Delta(X - z))$$

Note that we made use of the fact that, on  $C^2$ -functions, the usual Laplacian  $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$  can be rewritten as

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}}$$

in terms of the Pompeiu-Wirtinger derivatives

$$\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

In order to compute the Brown measure  $\mu_X$  of  $X$ , it is convenient to approximate  $\mu_X$  by certain regularizations  $\mu_{X,\varepsilon}$ . The **regularized Brown measures**  $\mu_{X,\varepsilon}$  are obtained by replacing in its defining equation the Fuglede-Kadison determinant  $\Delta$  by a certain regularization  $\Delta_\varepsilon$ . The **regularized Fuglede-Kadison determinant**  $\Delta_\varepsilon$  is given by

$$\Delta_\varepsilon(X) := \frac{1}{2} \phi(\log(XX^* + \varepsilon^2)).$$

Explicitly and again in the distributional sense, this means that

$$\mu_{X,\varepsilon}(z) = \frac{2}{\pi} \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \log(\Delta_\varepsilon(X - z)).$$

If we consider the **regularized Cauchy transform**

$$G_{X,\varepsilon}(z) = \phi((z - X)^*((z - X)(z - X)^* + \varepsilon^2)^{-1}),$$

which is a  $C^\infty$ -function on  $\mathbb{C}$  (but obviously not holomorphic), we may rewrite this as

$$(4.2) \quad d\mu_{X,\varepsilon}(z) = \frac{1}{\pi} \frac{\partial}{\partial \bar{z}} G_{X,\varepsilon}(z) d\lambda^2(z).$$

Since free probability theory – both in the scalar- and operator-valued setting – provides powerful tools to deal with Cauchy transforms, the formula for  $\mu_{X,\varepsilon}$  given in (4.2) looks rather appealing. However, there is the disadvantage that  $G_{X,\varepsilon}$  is fairly close to a usual Cauchy transform but still a totally different object.

Fortunately, we can calculate  $G_{X,\varepsilon}$  by using the so-called hermitian reduction method. This method is based on the  $M_2(\mathbb{C})$ -valued  $C^*$ -probability space  $(M_2(\mathcal{A}), E, M_2(\mathbb{C}))$ , where  $E$  denotes the conditional expectation that is given by  $\text{id}_{M_2(\mathbb{C})} \otimes \phi$ . We consider the selfadjoint element

$$\mathbb{X} := \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix} \in M_2(\mathcal{A}).$$

The regularized Cauchy transform  $G_{X,\varepsilon}(z)$  can then be obtained as the  $(2, 1)$ -entry of the  $M_2(\mathbb{C})$ -valued Cauchy transform of  $\mathbb{X}$ , if it is evaluated at the point

$$\Lambda_\varepsilon(z) := \begin{pmatrix} i\varepsilon & z \\ \bar{z} & i\varepsilon \end{pmatrix} \in \mathbb{H}^+(M_2(\mathbb{C})).$$

More precisely, we have for each  $z \in \mathbb{C}^+$  that

$$(4.3) \quad G_{X,\varepsilon}(z) = [G_{\mathbb{X}}(\Lambda_\varepsilon(z))]_{2,1}.$$

Collecting our observations, we see that we can compute the regularized Brown measures  $\mu_{X,\varepsilon}$  by (4.2) from its regularized Cauchy transforms  $G_{X,\varepsilon}$ , whereas the regularized Cauchy transform  $G_{X,\varepsilon}$  itself can be deduced by (4.3) from the  $M_2(\mathbb{C})$ -valued Cauchy transform of the selfadjoint element  $\mathbb{X}$ .

Hence, similarly to the case of selfadjoint polynomials, we end up with the purely algebraic question of finding appropriate representations of matrices  $X$  of rational expressions, which reduces the calculation of  $G_{\mathbb{X}}$  to the calculation of the operator-valued free additive convolution.



**4.5. Cauchy transforms of matrices of rational expressions.** Now, we want to discuss the important tool that bridges from scalar- to operator-valued free probability. Whereas this was done in [BMS13] by the method of linearization, we will work here with its generalization provided by formal linear representations, and in particular with the powerful machinery of realizations. This in turn will allow us to deal not only with polynomials but also with rational expressions, and in the case where they are regular at 0, we can do this even more in an optimal way by passing to minimal realizations. Moreover, the great flexibility of this theory allows to formulate our results in the more general situation of matrices of rational expressions. As usual, we will encode the information about distributions by (matrix-valued) Cauchy transforms.

**Be aware that the “NC” will be suppressed from now in any term like “NC rational expressions”, since we will work here solely in the noncommutative setting.**

But before giving the precise statement, we first recall that, according to Remark 4.3, any  $C^*$ -probability space  $(\mathcal{A}, \phi)$  gives rise to an operator-valued  $C^*$ -probability space  $(M_N(\mathcal{A}), E_N, M_N(\mathbb{C}))$  for each  $N \in \mathbb{N}$ . Here, we denote by  $E_N$  the conditional expectation which is induced by  $\text{id}_{M_N(\mathbb{C})} \otimes \phi$  under the usual identification  $M_N(\mathcal{A}) \cong M_N(\mathbb{C}) \otimes \mathcal{A}$ .

The first step is the following lemma in which we combine earlier results on the existence and uniqueness of minimal descriptor realizations of matrices of rational functions. For reasons of clarity, we first restrict ourselves to the case of (symmetric) real matrices, but in a subsequent remark, we will raise it to the complex case.

**Lemma 4.5.** *Let  $r$  be a symmetric  $k \times k$  matrix of rational expressions regular at 0 in formal variables  $x = (x_1, \dots, x_g)$ . Then  $r$  has a minimal symmetric descriptor realization*

$$(4.4) \quad \mathfrak{r}(x) = \Delta + \Xi^T (M_0 - M_1 x_1 - \dots - M_g x_g)^{-1} \Xi.$$

*with a symmetric  $k \times k$  matrix  $\Delta$ , symmetric  $N \times N$  matrices  $M_j$ ,  $j = 0, \dots, n$  with  $M_0^2 = I$ , and  $\Xi$  a  $N \times k$  matrix, and it is controllable and observable.*

*The representation (4.4) is unique in the sense that another such representation  $\tilde{\Delta}, \tilde{\Xi}, \tilde{M}_j$ ,  $j = 0, \dots, n$ , for the same  $r$  and with  $\tilde{\Delta} = \Delta$  satisfies*

$$(4.5) \quad S M_0 M_j = \tilde{M}_0 \tilde{M}_j S \quad S M_0 \Xi = \tilde{M}_0 \tilde{\Xi} \quad \Xi^T = \tilde{\Xi}^T S$$

*for some invertible matrix  $S$  satisfying  $S^T M_0 S = M_0$ .*

*Proof.* When  $x_1, \dots, x_g$  are symmetric non-commuting variables, existence of this minimal realization rephrases Lemma 3.4, 3. Uniqueness rephrases Lemma 3.8.  $\square$

*Remark 4.6.* The rational expression  $r$  can have complex coefficients and if it is selfadjoint, then (4.4) holds with matrices  $\Xi$  and selfadjoint matrices  $\Delta$  and  $M_j$ ,  $j = 1, \dots, n$ , having possibly complex entries. Here  $\Xi^T$ ,  $S^T$  become conjugate transpose  $\Xi^*$ ,  $S^*$ . This follows from the identification of complex



numbers  $z = a + ib$  with matrices in  $M_2(\mathbb{R})$  of the form

$$\eta_z := \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

Note  $\bar{z} = \eta_z^T$ . □

4.5.1. *Representation of Cauchy transforms.* Now, we are ready to formulate our main result on the representation of Cauchy transforms.

**Theorem 4.7.** *We consider any of the following situations:*

- (1) *Let  $r$  be a selfadjoint  $k \times k$  matrix of rational expressions in formal variables  $x = (x_1, \dots, x_g)$ . Consider any selfadjoint formal linear representation  $\rho = (Q, v)$  (whose existence is guaranteed by Theorem 5.14) and introduce an affine linear pencil  $\hat{\Lambda}$  by*

$$(4.6) \quad \hat{\Lambda}(x) := \begin{pmatrix} 0 & v^* \\ v & -Q(x) \end{pmatrix}.$$

*Furthermore, let  $X_1, \dots, X_g$  be selfadjoint elements in some  $C^*$ -probability space  $(\mathcal{A}, \phi)$ , such that the evaluation  $r(X)$  of  $r$  at  $X = (X_1, \dots, X_g)$  is well-defined (i.e., such that  $X \in \text{dom}_{\mathcal{A}}(r)$  is satisfied).*

- (2) *Let  $r$  be a selfadjoint  $k \times k$  matrix of rational expressions in formal variables  $x = (x_1, \dots, x_g)$ , which is regular at zero. Take any selfadjoint realization*

$$\mathfrak{r}(x) = \Delta + \Xi^*(M_0 - L_M(x))^{-1}\Xi$$

*of  $r$  with a linear pencil  $L_M(x) = M_1x_1 + \dots + M_gx_g$  consisting of selfadjoint  $N \times N$  matrices  $M_1, \dots, M_g$ , an invertible  $N \times N$  matrix  $M_0$  satisfying  $M_0^2 = I$ , a  $N \times k$  matrix  $\Xi$ , and a selfadjoint  $k \times k$  matrix  $\Delta$ . Define an affine linear pencil  $\hat{\Lambda}$  by*

$$(4.7) \quad \hat{\Lambda}(x) := \begin{pmatrix} \Delta & \Xi^* \\ \Xi & -(M_0 - L_M(x)) \end{pmatrix}.$$

*Furthermore, let  $X_1, \dots, X_g$  be selfadjoint elements in some  $C^*$ -probability space  $(\mathcal{A}, \phi)$  with a faithful tracial state  $\phi$ . Assume*

- (i) *either that the evaluation  $r(X)$  of  $r$  and the evaluation  $\mathfrak{r}(X)$  of  $\mathfrak{r}$  at  $X = (X_1, \dots, X_g)$  are both well-defined (i.e., such that  $X \in \text{dom}_{\mathcal{A}}(r) \cap \text{dom}_{\mathcal{A}}(\mathfrak{r})$  is satisfied),*
- (ii) *or that  $r$  is minimal and its evaluation  $r(X)$  at  $X = (X_1, \dots, X_g)$  is well-defined (i.e., such that  $X \in \text{dom}_{\mathcal{A}}(r)$  is satisfied).*

*Then, in each of these situations the following holds true: For all  $B \in \mathbb{H}^+(M_k(\mathbb{C}))$ , we have that*

$$(4.8) \quad \begin{aligned} & (B \otimes I_{\mathcal{A}} - r(X))^{-1} \\ &= \begin{pmatrix} I_k \otimes I_{\mathcal{A}} & 0 \end{pmatrix} \left( \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \otimes I_{\mathcal{A}} - \hat{\Lambda}(X) \right)^{-1} \begin{pmatrix} I_k \otimes I_{\mathcal{A}} \\ 0 \end{pmatrix}. \end{aligned}$$

*In particular, the  $M_k(\mathbb{C})$ -valued Cauchy transform  $G_{r(X_1, \dots, X_g)}$  (calculated with respect to the conditional expectation  $E_k$ ) is determined by the  $M_{N+k}(\mathbb{C})$ -valued*

Cauchy transform  $G_{\hat{\Lambda}(X_1, \dots, X_g)}$  (calculated with respect to the conditional expectation  $E_{N+k}$ ) by

$$(4.9) \quad G_{r(X_1, \dots, X_g)}(B) = \lim_{\varepsilon \searrow 0} \begin{pmatrix} I_k & 0 \\ 0 & i\varepsilon I_N \end{pmatrix} G_{\hat{\Lambda}(X_1, \dots, X_g)} \left( \begin{pmatrix} B & 0 \\ 0 & i\varepsilon I_N \end{pmatrix} \right) \begin{pmatrix} I_k \\ 0 \end{pmatrix}.$$

for all  $B \in \mathbb{H}^+(M_k(\mathbb{C}))$ .

*Proof.* First of all, we will prove that the conclusion of the Theorem holds true, not only in the described situations (1), (2) (i), and (2) (ii), but in fact under the following more general assumptions:

Let  $r$  be a selfadjoint  $k \times k$  matrix of rational expressions in the formal variables  $x_1, \dots, x_g$  and let  $X_1, \dots, X_g$  be selfadjoint elements in any non-commutative  $C^*$ -probability space  $(\mathcal{A}, \phi)$ , such that  $X = (X_1, \dots, X_g) \in \text{dom}_{\mathcal{A}}(r)$ . Assume that there is a  $N \times N$  NC linear pencil

$$\Lambda_M(x) = M_0 - L_M(x) \quad \text{with} \quad L_M(X) = M_1 x_1 + \dots + M_g x_g,$$

where  $M_0, M_1, \dots, M_g$  are selfadjoint  $N \times N$  matrices,  $\Xi$  is a  $N \times k$  matrix, and  $\Delta$  is a selfadjoint  $k \times k$  matrix, such that  $\Lambda_M(X) \in M_N(\mathcal{A})$  is invertible and

$$r(X) = \Delta + \Xi^* \Lambda_M(X)^{-1} \Xi$$

holds true. Put

$$\hat{\Lambda}(x) := \begin{pmatrix} \Delta & \Xi^* \\ \Xi & -\Lambda_M(X) \end{pmatrix}.$$

Note that we allow implicitly that the data  $(\Delta, \Lambda_M, \Xi)$  depend on the concrete choice variables  $X$  in  $(\mathcal{A}, \phi)$ .

For proving that these assumptions give the desired conclusion, we proceed as follows: Abbreviate  $\begin{pmatrix} I_k \\ 0 \end{pmatrix}$  to  $\mathcal{E}_k$  and  $\mathcal{E}_k \otimes I_{\mathcal{A}}$  to  $\rho_{\mathcal{A}}$ , that is

$$\mathcal{E}_k = \begin{pmatrix} I_k \\ 0 \end{pmatrix} \quad \text{and} \quad \rho_{\mathcal{A}} = \mathcal{E}_k \otimes I_{\mathcal{A}}.$$

Since  $\Lambda_M(X) = M_0 - L_M(X)$  is invertible in  $M_N(\mathcal{A})$  by assumption, the well-known Schur complement formula tells us, for any  $B \in M_k(\mathbb{C})$ , that the matrix

$$\begin{pmatrix} B - \Delta & -\Xi^* \\ -\Xi & M_0 - L_M(X) \end{pmatrix}$$

is invertible in  $M_{N+k}(\mathcal{A})$  if and only if its Schur complement

$$B - (\Delta + \Xi^*(M_0 - L_M(X))^{-1}\Xi) = B - r(X)$$

is invertible in  $M_k(\mathcal{A})$ . Hence, since  $r(X)$  is selfadjoint and thus  $B - r(X)$  must be invertible at least for all  $B \in \mathbb{H}^+(M_k(\mathbb{C}))$ , we know that

$$\begin{pmatrix} B - \Delta & -\Xi^* \\ -\Xi & M_0 - L_M(X) \end{pmatrix}$$

is invertible for all  $B \in \mathbb{H}^+(M_k(\mathbb{C}))$ . Furthermore, the Schur complement formula yields in this case that

$$\begin{aligned} \mathcal{E}_k^* \left( \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} - \hat{\Lambda}(x) \right)^{-1} \mathcal{E}_k &= \mathcal{E}_k^* \begin{pmatrix} B - \Delta & -\Xi^* \\ -\Xi & M_0 - L_M(X) \end{pmatrix}^{-1} \mathcal{E}_k \\ &= (B - r(X))^{-1}, \end{aligned}$$

Thus for each  $B \in \mathbb{H}^+(M_k(\mathbb{C}))$

$$(\mathcal{E}_k \otimes I_A)^* \left( \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \otimes I_A - \hat{\Lambda}(X) \right)^{-1} (\mathcal{E}_k \otimes I_A) = (B \otimes I_A - r(X))^{-1},$$

which gives the stated formula (4.8).

For seeing (4.9), we first note that by definition

$$E_k[\rho_{\mathcal{A}}^* W \rho_{\mathcal{A}}] = \mathcal{E}_k^* E_{N+k}[W] \mathcal{E}_k \quad \text{for any } W \in M_{N+k}(\mathcal{A}).$$

Thus, we get by applying  $E_k$  to both sides of (4.8) that

$$\mathcal{E}_k^* E_{N+k} \left[ \left( \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \otimes I_A - \hat{\Lambda}(X) \right)^{-1} \right] \mathcal{E}_k = E_k[(B \otimes I_A - r(X))^{-1}]$$

and hence, by definition of  $G_{r(X_1, \dots, X_g)}$ , that

$$(4.10) \quad \mathcal{E}_k^* E_{N+k} \left[ \left( \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \otimes I_A - \hat{\Lambda}(X) \right)^{-1} \right] \mathcal{E}_k = G_{r(X_1, \dots, X_g)}(B).$$

Now, we encounter the annoying fact that the expression

$$E_{N+k} \left[ \left( \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \otimes I_A - \hat{\Lambda}(X) \right)^{-1} \right]$$

appearing on the left hand side of this equation is not precisely an evaluation of the  $M_{N+k}(\mathbb{C})$ -valued Cauchy transform of  $\hat{\Lambda}(X_1, \dots, X_g)$  but rather a boundary value of it, since  $\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$  does not belong to the upper half-plane  $\mathbb{H}^+(M_{N+k}(\mathbb{C}))$ . The representation given in (4.9) therefore involves a limit procedure which allows to move our observation by  $\begin{pmatrix} B & 0 \\ 0 & i\varepsilon I_N \end{pmatrix}$  to the domain of the Cauchy transforms, where all our regular tools apply.

In order to convince ourselves that the representation given in (4.9) is indeed correct, we just have to observe that  $G_{\hat{\Lambda}(X_1, \dots, X_g)}$  can be extended in the obvious way by

$$G_{\hat{\Lambda}(X_1, \dots, X_g)}(A) = E_{N+k}[(A \otimes I_A - \hat{\Lambda}(X_1, \dots, X_g))^{-1}]$$

to the open set  $\Omega \supset \mathbb{H}^+(M_{N+k}(\mathbb{C}))$  of all matrices  $A \in M_{N+k}(\mathbb{C})$ , for which  $A \otimes I_A - \hat{\Lambda}(X_1, \dots, X_g)$  is invertible in  $M_{N+k}(\mathcal{A})$ . Since this extension is regular

and in particular continuous, and since  $\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \in \Omega$ , we deduce that

$$(4.11) \quad \begin{aligned} & \lim_{\varepsilon \searrow 0} G_{\hat{\Lambda}(X_1, \dots, X_g)} \left( \begin{pmatrix} B & 0 \\ 0 & i \in I_N \end{pmatrix} \right) \\ &= E_{N+k} \left[ \left( \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \otimes I_{\mathcal{A}} - \hat{\Lambda}(X) \right)^{-1} \right]. \end{aligned}$$

By the continuity of the map compressing  $M_{N+k}(\mathbb{C})$  to  $M_k(\mathbb{C})$ , a combination of (4.10) and (4.11) yields the stated formula (4.9).

Finally, it only remains to prove that all situations (1), (2) (i), and (2) (ii) fit into this general frame.

- (1) Since  $X = X^*$  belongs to  $\text{dom}_{\mathcal{A}}^{\text{sa}}(r)$ , we know according to definition 5.13 that  $X$  also belongs to  $\text{dom}_{\mathcal{A}}(Q^{-1})$  for any given selfadjoint matrix-valued formal linear representation  $\rho = (Q, v)$  of  $r$  and that it moreover satisfies

$$r(X) = -uQ(X)^{-1}v.$$

- (2) First of all, we note that the additional assumption on  $\phi$  being a faithful trace guarantees according to Lemma 2.2 that  $\mathcal{A}$  is stably finite.

- (i) Since  $X \in \text{dom}_{\mathcal{A}}(r) \cap \text{dom}_{\mathcal{A}}(\mathfrak{r})$ , Corollary 6.2 tells us that the values  $r(X)$  and  $\mathfrak{r}(X)$  coincide, i.e.

$$r(X) = \Delta + \Xi^*(M_0 - L_M(X))^{-1}\Xi.$$

- (ii) Since  $X = X^*$  belongs to  $\text{dom}_{\mathcal{A}}^{\text{sa}}(r)$ , Theorem 5.16 tells us that  $X$  belongs to the domain  $\text{dom}_{\mathcal{A}}(\mathfrak{r})$  of our given minimal realization  $\mathfrak{r}$  of  $r$  and that the evaluation of  $r$  and  $\mathfrak{r}$  at  $X$  yields the same result, i.e.

$$r(X) = \Delta + \Xi^*(M_0 - L_M(X))^{-1}\Xi.$$

This concludes the proof.  $\square$

4.5.2. *Uniqueness of the representation.* In Theorem 4.7, we constructed for a given realization

$$\mathfrak{r}(x) = \Delta + \Xi^*(M_0 - M_1x_1 - \dots - M_gx_g)^{-1}\Xi$$

of  $r(x)$  a matrix  $\hat{\Lambda}(x)$  by (4.7), i.e.

$$\hat{\Lambda}(x) = \begin{pmatrix} \Delta & \Xi^* \\ \Xi & -(M_0 - L_M(x)) \end{pmatrix},$$

which can be seen as a shifted version of the linearizations considered in [BMS13].

The uniqueness result formulated in Lemma 4.5 for minimal descriptor realizations of  $r(x)$  guarantees that matrices  $\hat{\Lambda}(x)$  and  $\tilde{\hat{\Lambda}}(x)$  obtained from different

minimal descriptor realizations of  $r(x)$  with the same feed through term  $\Delta$  are related by  $\hat{\Lambda}(x) = \hat{S}^* \tilde{\Lambda}(x) \hat{S}$ , where the matrix  $\hat{S}$  is of the form

$$\hat{S} = \begin{pmatrix} I_k & 0 \\ 0 & S \end{pmatrix}$$

for some invertible  $N \times N$  matrix  $S$ .

Indeed, if we consider two descriptor realizations

$$\mathfrak{r}(x) = \Delta + \Xi^*(M_0 - L_M(x))^{-1}\Xi \quad \text{and} \quad \tilde{\mathfrak{r}}(x) = \Delta + \tilde{\Xi}^*(\tilde{M}_0 - L_{\tilde{M}}(x))^{-1}\tilde{\Xi}$$

of  $r$ , which both satisfy the minimality constraint, we know from Lemma 4.5 that there is an invertible  $N \times N$  matrix  $S$  satisfying  $S^* \tilde{M}_0 S = M_0$ , such that

$$(4.12) \quad SM_0 M_j = \tilde{M}_0 \tilde{M}_j S, \quad SM_0 \Xi = \tilde{M}_0 \tilde{\Xi}, \quad \Xi^* = \tilde{\Xi}^* S.$$

Hence, the matrices

$$\hat{\Lambda}(x) = \begin{pmatrix} \Delta & \Xi^* \\ \Xi & -(M_0 - L_M(x)) \end{pmatrix} \quad \text{and} \quad \tilde{\Lambda}(x) = \begin{pmatrix} \Delta & \tilde{\Xi}^* \\ \tilde{\Xi} & -(\tilde{M}_0 - L_{\tilde{M}}(x)) \end{pmatrix},$$

which we constructed according to Theorem 4.7, are related by

$$\hat{\Lambda}(x) = \hat{S}^* \tilde{\Lambda}(x) \hat{S}, \quad \text{where} \quad \hat{S} := \begin{pmatrix} I_k & 0 \\ 0 & S \end{pmatrix}.$$

Indeed, we can check that

$$\hat{S}^* \tilde{\Lambda}(x) \hat{S} = \begin{pmatrix} 0 & \tilde{\Xi}^* S \\ S^* \tilde{\Xi} & -S^*(\tilde{M}_0 - L_{\tilde{M}}(x))S \end{pmatrix} = \begin{pmatrix} \Delta & \Xi^* \\ \Xi & -(M_0 - L_M(x)) \end{pmatrix} = \hat{\Lambda}(x),$$

since it holds true that

- $\tilde{\Xi}^* S = \Xi^*$  and therefore also  $S^* \tilde{\Xi} = \Xi$ ;
- $S^* \tilde{M}_0 S = M_0$  and  $S^* \tilde{M}_j S = M_j$ .

The formula  $S^* \tilde{M}_j S = M_j$  stated above can be shown as follows: The assumption  $SM_0 M_j = \tilde{M}_0 \tilde{M}_j S$  can be rewritten as  $M_j = (M_0 S^{-1} \tilde{M}_0) \tilde{M}_j S$  and since by construction  $S^* \tilde{M}_0 S = M_0$  holds, which gives  $S^* = M_0 S^{-1} \tilde{M}_0$ , we finally get the stated relation  $S^* \tilde{M}_j S = M_j$ .

**4.6. How to calculate distributions and Brown measures of rational expressions.** Theorem 4.7 allows us to solve the following problems, which we presented in the previous subsections.

**Problem 4.8.** *Given a selfadjoint rational expression  $r$  in formal variables  $x = (x_1, \dots, x_g)$ . Let  $X_1, \dots, X_g$  be freely independent selfadjoint elements in some non-commutative  $C^*$ -probability space  $(\mathcal{A}, \phi)$  for which the evaluation  $r(X_1, \dots, X_g)$  is well-defined. If the distribution of each of the  $X_j$ 's is known, how can we compute the distribution of  $r(X_1, \dots, X_g)$ ?*

**Problem 4.9.** *Given an arbitrary rational expression  $r$  in formal variables  $x = (x_1, \dots, x_g)$ . Let  $X_1, \dots, X_g$  be freely independent selfadjoint elements in some tracial  $W^*$ -probability space  $(\mathcal{A}, \phi)$  for which the evaluation  $r(X_1, \dots, X_g)$  is well-defined. If the distribution of each of the  $X_j$ 's is known, how can we compute the Brown-measure of  $r(X_1, \dots, X_g)$ ?*

4.6.1. *An algorithmic solution of Problem 4.8.* Let us first discuss Problem 4.8. If we apply Theorem 4.7 in the case  $k = 1$  to  $r$ , we observe, according to (4.9), that the scalar-valued Cauchy transform of  $r(X_1, \dots, X_g)$  is determined by the  $M_{N+1}(\mathbb{C})$ -valued Cauchy transform  $\hat{\Lambda}(X_1, \dots, X_g)$  for some (affine) linear pencil

$$\hat{\Lambda}(x) = \hat{\Lambda}_0 + \hat{\Lambda}_1 x_1 + \dots + \hat{\Lambda}_g x_g$$

with selfadjoint matrices  $\hat{\Lambda}_0, \hat{\Lambda}_1, \dots, \hat{\Lambda}_g \in M_{N+1}(\mathbb{C})$ . In fact, we have

$$G_{r(X_1, \dots, X_g)}(z) = \lim_{\varepsilon \searrow 0} (1 \quad 0) G_{\hat{\Lambda}(X_1, \dots, X_g)} \left( \begin{pmatrix} z & 0 \\ 0 & i\varepsilon I_N \end{pmatrix} \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

for each  $z \in \mathbb{C}^+$ . Note that, depending on the given situation, we can construct  $\hat{\Lambda}$  either from any selfadjoint formal linear representations or from suitable realizations, in particular the minimal realizations, where the latter clearly has the great computational advantage of reducing the dimension of  $\hat{\Lambda}$ .

Since

$$\hat{\Lambda}(X_1, \dots, X_g) = \hat{\Lambda}_0 + \hat{\Lambda}_1 X_1 + \dots + \hat{\Lambda}_g X_g,$$

where the matrices  $\hat{\Lambda}_1 X_1, \dots, \hat{\Lambda}_g X_g$  are, according to Remark 4.3, free with amalgamation over  $M_{N+1}(\mathbb{C})$ , the  $M_{N+1}(\mathbb{C})$ -valued Cauchy transform of  $\hat{\Lambda}(X_1, \dots, X_g)$  can be computed by means of Theorem 4.4.

The desired distribution of  $r(X_1, \dots, X_g)$  is then obtained by Stieltjes inversion formula.

4.6.2. *An algorithmic solution of Problem 4.9.* Finally, let us discuss the algorithmic solution of Problem 4.9. If we apply Theorem 4.7, in the case  $k = 2$ , to the selfadjoint matrix-valued rational expression

$$(4.13) \quad \hat{r}(x) := \begin{pmatrix} 0 & r(x) \\ r^*(x) & 0 \end{pmatrix},$$

we obtain some (affine) linear pencil

$$\hat{\Lambda}(x) = \hat{\Lambda}_0 + \hat{\Lambda}_1 x_1 + \dots + \hat{\Lambda}_g x_g$$

with selfadjoint matrices  $\hat{\Lambda}_0, \hat{\Lambda}_1, \dots, \hat{\Lambda}_g \in M_{N+2}(\mathbb{C})$ , such that the  $M_2(\mathbb{C})$ -valued Cauchy transform of  $\mathfrak{r}(X_1, \dots, X_g)$  is determined by the  $M_{N+2}(\mathbb{C})$ -valued Cauchy transform of  $\hat{\Lambda}(X_1, \dots, X_g)$ . Indeed, following (4.9), we have

$$G_{\mathfrak{r}(X_1, \dots, X_g)}(B) = \lim_{\varepsilon \searrow 0} (I_2 \quad 0) G_{\hat{\Lambda}(X_1, \dots, X_g)} \left( \begin{pmatrix} B & 0 \\ 0 & i\varepsilon I_N \end{pmatrix} \right) \begin{pmatrix} I_2 \\ 0 \end{pmatrix}$$

for each  $B \in \mathbb{H}^+(M_2(\mathbb{C}))$ . As before in Problem 4.8, we note that Theorem 4.7 allows us, in the case where  $r$  (and hence  $\hat{r}$ ) is regular at 0, to construct  $\hat{\Lambda}$  from any minimal realization of  $\hat{r}$ . However, Theorem 4.7 covers another important

situation: if some realization of  $r$ , whose domain contains  $(X_1, \dots, X_g)$ , is already given, we can use this according to Lemma 4.10 below to construct directly a selfadjoint realization of  $\hat{r}$ , whose domain contains  $(X_1, \dots, X_g)$  as well.

Since

$$\hat{\Lambda}(X_1, \dots, X_g) = \hat{\Lambda}_0 + \hat{\Lambda}_1 X_1 + \dots + \hat{\Lambda}_g X_g,$$

where the matrices  $\hat{\Lambda}_1 X_1, \dots, \hat{\Lambda}_g X_g$  are according to Remark 4.3 free with amalgamation over  $M_{N+2}(\mathbb{C})$ , the  $M_{N+2}(\mathbb{C})$ -valued Cauchy transform of  $\hat{\Lambda}(X_1, \dots, X_g)$  can be computed by means of Theorem 4.4.

The regularized Cauchy transform  $G_{r(X_1, \dots, X_g), \varepsilon}$  is then determined by (4.3), i.e. we have

$$G_{r(X_1, \dots, X_g), \varepsilon}(z) = [G_{\hat{r}(X_1, \dots, X_g)}(\Lambda_\varepsilon(z))]_{2,1}, \quad z \in \mathbb{C}^+.$$

Recall that  $\Lambda_\varepsilon(z) = \begin{pmatrix} i\varepsilon & z \\ \bar{z} & i\varepsilon \end{pmatrix}$ . The regularized Brown measure  $\mu_{r(X_1, \dots, X_g), \varepsilon}$ , which approximates the desired Brown measure as  $\varepsilon \searrow 0$ , can then be obtained, according to (4.2), from the regularized Cauchy transform by

$$d\mu_{r(X_1, \dots, X_g), \varepsilon}(z) = \frac{1}{\pi} \frac{\partial}{\partial \bar{z}} G_{r(X_1, \dots, X_g), \varepsilon}(z) d\lambda^2(z).$$

We conclude by the useful observation that a selfadjoint realization of the matrix  $\hat{r}$ , which we introduced above in (4.13), can be constructed from any realization of the involved rational expression  $r$ . The precise statement, which in addition covers the case of rational expressions, which are not necessarily regular at 0, reads as follows.

**Lemma 4.10.** *Let  $r$  be a scalar-valued rational expression in the formal variables  $x = (x_1, \dots, x_g)$  and consider as in (4.13) the matrix-valued rational expression  $\hat{r}$  given by*

$$\hat{r} := \begin{pmatrix} 0 & r \\ r^* & 0 \end{pmatrix}.$$

(1) *If  $\rho = (u, Q, v)$  is any formal linear representation of  $r$ , then*

$$\hat{\rho} = (\hat{Q}, \hat{v}) := \left( \begin{pmatrix} 0 & Q \\ Q^* & 0 \end{pmatrix}, \begin{pmatrix} 0 & v \\ u^* & 0 \end{pmatrix} \right)$$

*gives a selfadjoint formal linear representation of  $\hat{r}$ .*

(2) *If  $r$  is regular at 0 and if*

$$\mathbf{r}(x) = \Delta + \Xi_1(M_0 - L_M(x))^{-1}\Xi_2$$

*with  $L_M(x) = M_1 x_1 + \dots + M_g x_g$  is any descriptor realization of  $r$ , then we may obtain a selfadjoint realization of  $\hat{r}$  by*

$$\hat{\mathbf{r}}(x) = \begin{pmatrix} 0 & \Delta \\ \Delta^* & 0 \end{pmatrix} + \hat{\Xi}^*(\hat{M}_0 - L_{\hat{M}}(x))^{-1}\hat{\Xi},$$

*where we put  $\hat{\Xi} := \begin{pmatrix} 0 & \Xi_2 \\ \Xi_1^* & 0 \end{pmatrix}$  and  $\hat{M}_j := \begin{pmatrix} 0 & M_j \\ M_j^* & 0 \end{pmatrix}$  for  $j = 0, 1, \dots, g$ .*



*Proof.* (1) Let  $\mathcal{A}$  be any  $*$ -algebra. We clearly have that  $\text{dom}_{\mathcal{A}}(Q^{-1}) = \text{dom}_{\mathcal{A}}(\hat{Q}^{-1})$ , and since  $\rho$  is a formal linear representation of  $r$ , we have by definition  $\text{dom}_{\mathcal{A}}(r) \subseteq \text{dom}_{\mathcal{A}}(Q^{-1})$ . In combination, this gives  $\text{dom}_{\mathcal{A}}(r) \subseteq \text{dom}_{\mathcal{A}}(\hat{Q}^{-1})$  and in particular  $\text{dom}_{\mathcal{A}}^{\text{sa}}(r) \subseteq \text{dom}_{\mathcal{A}}(\hat{Q}^{-1})$ . Furthermore,  $\rho$  enjoys the property that  $r(X) = -uQ(X)^{-1}v$  and hence  $r^*(X^*) = -v^*Q^*(X^*)^{-1}u^*$  holds for any  $X \in \text{dom}_{\mathcal{A}}(r)$ . Thus, if we take  $X \in \text{dom}_{\mathcal{A}}^{\text{sa}}(r)$ , we may deduce that  $r(X) = -uQ(X)^{-1}v$  and  $r^*(X) = -v^*Q^*(X)^{-1}u^*$  holds, so that

$$\begin{aligned}
-\hat{v}^*\hat{Q}(X)^{-1}\hat{v} &= -\begin{pmatrix} 0 & u \\ v^* & 0 \end{pmatrix} \begin{pmatrix} 0 & Q(X) \\ Q^*(X) & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & v \\ u^* & 0 \end{pmatrix} \\
&= -\begin{pmatrix} 0 & u \\ v^* & 0 \end{pmatrix} \begin{pmatrix} 0 & Q^*(X)^{-1} \\ Q(X)^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & v \\ u^* & 0 \end{pmatrix} \\
&= -\begin{pmatrix} 0 & u \\ v^* & 0 \end{pmatrix} \begin{pmatrix} Q^*(X)^{-1}u^* & 0 \\ 0 & Q(X)^{-1}v \end{pmatrix} \\
&= -\begin{pmatrix} 0 & uQ(X)^{-1}v \\ v^*Q^*(X)^{-1}u^* & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & r(X) \\ r^*(X) & 0 \end{pmatrix} \\
&= \hat{r}(X).
\end{aligned}$$

This shows that  $\hat{\rho}$  is indeed a selfadjoint formal linear representation.

(2) First of all, we note that  $\hat{M}_0^* = \hat{M}_0$  and  $\hat{M}_0^2 = I_{2k}$ , since by assumption  $M_0^* = M_0$  and  $M_0^2 = I_k$  holds. Thus,  $\hat{r}$  is indeed a selfadjoint descriptor realization of some matrix-valued rational expression. It remains to prove that it forms in fact a realization of  $\hat{r}$ . For this purpose, it is according to [HMOV06, Proposition A.7] sufficient to check that  $\hat{r}$  and  $\hat{r}$  are  $\mathbb{M}(\mathbb{C})_{\text{sa}}$ -evaluation equivalent.

For doing so, we take any matrix  $X \in \mathbb{M}(\mathbb{C})_{\text{sa}}$ , which belongs to the domain of  $\hat{r}$  and to the domain of  $\hat{r}$ . Since

$$\hat{M}_0 - L_{\hat{M}}(X) = \begin{pmatrix} 0 & M_0 - L_M(X) \\ M_0^* - L_{M^*}(X) & 0 \end{pmatrix},$$

we have that  $X$  also belongs to the domain of  $r$  and furthermore

$$\begin{aligned}
&\hat{\Xi}^*(\hat{M}_0 - L_{\hat{M}}(X))^{-1}\hat{\Xi} \\
&= \begin{pmatrix} 0 & \Xi_2 \\ \Xi_1^* & 0 \end{pmatrix}^* \begin{pmatrix} 0 & M_0 - L_M(X) \\ M_0^* - L_{M^*}(X) & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & \Xi_2 \\ \Xi_1^* & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \Xi_1 \\ \Xi_2^* & 0 \end{pmatrix} \begin{pmatrix} 0 & (M_0^* - L_{M^*}(X))^{-1} \\ (M_0 - L_M(X))^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \Xi_2 \\ \Xi_1^* & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \Xi_1 \\ \Xi_2^* & 0 \end{pmatrix} \begin{pmatrix} (M_0^* - L_{M^*}(X))^{-1}\Xi_1^* & 0 \\ 0 & (M_0 - L_M(X))^{-1}\Xi_2 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \Xi_1(M_0 - L_M(X))^{-1}\Xi_2 \\ \Xi_2^*(M_0^* - L_{M^*}(X))^{-1}\Xi_1^* & 0 \end{pmatrix},
\end{aligned}$$

so that

$$\hat{\mathfrak{r}}(X) = \begin{pmatrix} 0 & \Delta \\ \Delta^* & 0 \end{pmatrix} + \hat{\Xi}^*(\hat{M}_0 - L_{\hat{M}}(X))^{-1}\hat{\Xi} = \begin{pmatrix} 0 & \mathfrak{r}(X) \\ \mathfrak{r}(X)^* & 0 \end{pmatrix}.$$

Moreover, since  $\mathfrak{r}$  is a realization of  $r$  and therefore  $\mathfrak{r}(X) = r(X)$  holds, we may continue

$$\hat{\mathfrak{r}}(X) = \begin{pmatrix} 0 & \mathfrak{r}(X) \\ \mathfrak{r}(X)^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & r(X) \\ r(X)^* & 0 \end{pmatrix} = \hat{r}(X).$$

This concludes the proof.  $\square$

A more complicated construction underlies the minimal symmetric realization asserted in Lemma 3.4 (3).

**4.7. Examples.** We conclude with several concrete examples by which we discuss the theory presented above. Whereas the Examples 4.11, 4.12 and 4.14 concern Problem 4.8 for the anticommutator  $x_1x_2 + x_2x_1$ , for the commutator  $i(x_1x_2 - x_2x_1)$ , and finally for the much more complicated rational expression determined by its representation

$$\begin{pmatrix} \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 - \frac{1}{4}x_1 & -\frac{1}{4}x_2 \\ -\frac{1}{4}x_2 & 1 - \frac{1}{4}x_1 \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix},$$

Example 4.15 concerns Problem 4.9 for the rational expression given by

$$\begin{pmatrix} 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 - \frac{1}{4}x_1 & -ix_2 \\ -\frac{1}{4}x_2 & 1 - \frac{1}{4}x_1 \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}.$$

Since we will perform the numerical computations in the case of freely independent elements  $X_1, \dots, X_g$  whose distribution is given either by the semicircular distribution or by the free Poisson distribution (where the latter is also known as the Marchenko-Pastur distribution), their joint distribution arises also as the limit of the joint distribution of (classically) independent Gaussian and Wishart random matrices  $(X_1^{(N)}, \dots, X_g^{(N)})$  of dimension  $N \times N$ , respectively, in the limit  $N \rightarrow \infty$ .

Thus, we will compare below the computed distribution of  $r(X_1, \dots, X_g)$  with the normalized histogram of all eigenvalues of the random matrix obtained by  $r(X_1^{(N)}, \dots, X_g^{(N)})$ . This will show in all cases a nice conformity.

Note that, whereas in the special case of a non-commutative polynomial  $r$  the convergence of the eigenvalue distribution of  $r(X_1^{(N)}, \dots, X_g^{(N)})$  to  $r(X_1, \dots, X_g)$  is obvious, the situation for general  $r$  is more intricate. The main difficulty here is that  $(X_1^{(N)}, \dots, X_g^{(N)})$  must belong almost surely to the domain of  $r$  if their dimension  $N$  is sufficiently large. Nevertheless, it is conceivable that this can be checked in many cases.

In the Brown measure case, however, despite the amazing similarity between the output of our algorithm and of the random matrix simulation, there is up to now no general statement which would give a rigorous justification of this phenomenon.

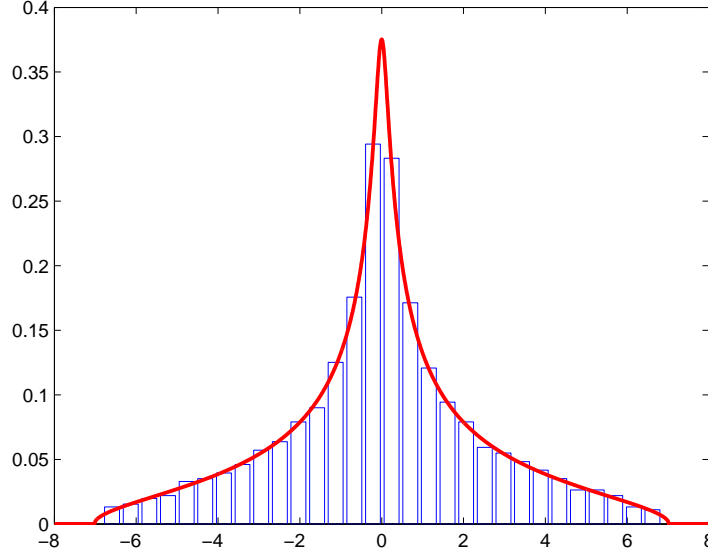


FIGURE 1. Histogram of eigenvalues of  $p(X_1^{(N)}, X_2^{(N)})$ , where  $p(x_1, x_2)$  was defined in Example 4.11, for one realization of independent random matrices  $X_1^{(N)}, X_2^{(N)}$ , where  $X_1^{(N)}$  is a Wishart random matrix and  $X_2^{(N)}$  a Gaussian random matrix, both of size  $N = 1000$ , compared with the distribution of  $p(X_1, X_2)$  for freely independent elements  $X_1, X_2$ , where  $X_1$  is a free Poisson element and  $X_2$  a semicircular element.

*Example 4.11* (Anti-commutator, see Figure 1). We consider the **anti-commutator**

$$p(x_1, x_2) := x_1 x_2 + x_2 x_1.$$

In order to produce a (selfadjoint) representation and finally a (selfadjoint) realization of  $p$ , we could of course just apply the algorithm that we will presented in detail in Section 5. However, since we can write

$$p(x_1, x_2) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

this gives directly the selfadjoint representation

$$p(x_1, x_2) = - \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & x_1 & x_2 & -1 \\ x_1 & 0 & -1 & 0 \\ x_2 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

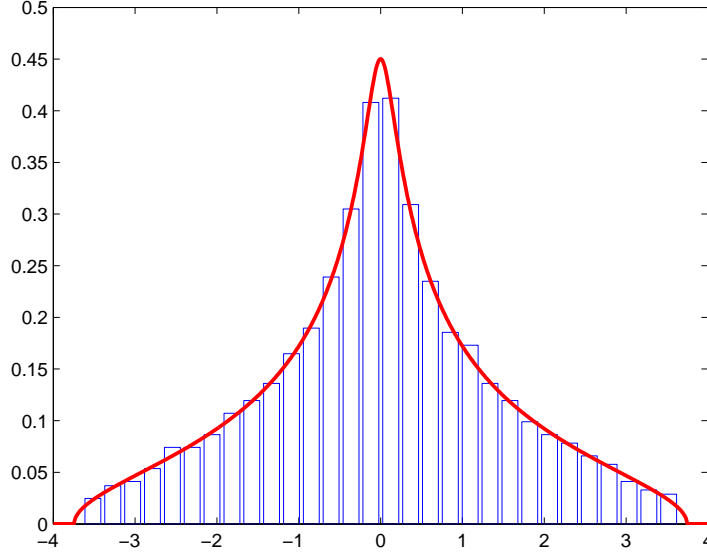


FIGURE 2. Histogram of eigenvalues of  $q(X_1^{(N)}, X_2^{(N)})$ , where the polynomial  $q(x_1, x_2)$  was defined in Example 4.12, for one realization of independent random matrices  $X_1^{(N)}, X_2^{(N)}$ , where  $X_1^{(N)}$  is a Wishart random matrix and  $X_2^{(N)}$  a Gaussian random matrix, both of size  $N = 1000$ , compared with the distribution of  $q(X_1, X_2)$  for freely independent elements  $X_1, X_2$ , where  $X_1$  is a free Poisson element and  $X_2$  a semicircular element.

According to Theorem 4.7, we consider now the matrix

$$\hat{\Lambda}(x_1, x_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & x_1 & x_2 & -1 \\ 0 & x_1 & 0 & -1 & 0 \\ 0 & x_2 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \end{pmatrix}$$

which decomposes as  $\hat{\Lambda}(x_1, x_2) = \hat{\Lambda}_0 + \hat{\Lambda}_1 x_1 + \hat{\Lambda}_2 x_2$ , where

$$\hat{\Lambda}_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{\Lambda}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \hat{\Lambda}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

□

*Example 4.12* (Commutator, see Figure 2). We consider now the **commutator**

$$q(x_1, x_2) := i(x_1x_2 - x_2x_1).$$

Like for the anti-commutator in Example 4.11 above, we can use the similar decomposition

$$q(x_1, x_2) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

in order to construct a selfadjoint representation of  $q$  by

$$p(x_1, x_2) = - \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & x_1 & x_2 & -1 \\ x_1 & 0 & i & 0 \\ x_2 & -i & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

We may apply Theorem 4.7, which tells us that we should consider

$$\hat{\Lambda}(x_1, x_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & x_1 & x_2 & -1 \\ 0 & x_1 & 0 & i & 0 \\ 0 & x_2 & -i & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \end{pmatrix}$$

which decomposes as  $\hat{\Lambda}(x_1, x_2) = \hat{\Lambda}_0 + \hat{\Lambda}_1x_1 + \hat{\Lambda}_2x_2$ , where

$$\hat{\Lambda}_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & -i & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{\Lambda}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \hat{\Lambda}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

□

The shortcut that was used in both of the previous examples in order to produce realizations without using the algorithm of Section 5 relies on a more general observation, which we include here for the seek of completeness. This is the content of the following remark.

*Remark 4.13.* Assume that  $Q_1, \dots, Q_k$  are rectangular matrices of rational expressions in the formal variables  $x_1, \dots, x_g$ , such that  $Q_j$  is a  $(n_j \times n_{j+1})$ -matrix for  $j = 1, \dots, k$ , where  $n_1, \dots, n_{k+1}$  are some positive integers. Furthermore, let  $\Xi_j \in M_{n_j}(\mathbb{C})$ , for  $j = 1, \dots, k+1$ , be invertible matrices. Consider now the  $(n_1 \times n_{k+1})$ -matrix

$$r := \Xi_1^{-1} Q_1 \Xi_2^{-1} Q_2 \cdots \Xi_k^{-1} Q_k \Xi_{k+1}^{-1}$$

of rational expression and introduce a matrix  $Q$  of size  $n := n_1 + \dots + n_k$  by

$$Q := \begin{pmatrix} & & & Q_1 & -\Xi_1 \\ & & & \ddots & -\Xi_2 \\ & & Q_{k-1} & \ddots & \\ Q_k & & -\Xi_k & & \\ -\Xi_{k+1} & & & & \end{pmatrix}.$$

It is then not hard to check inductively that we have for any unital complex algebra  $\mathcal{A}$  and for any  $(X_1, \dots, X_g) \in \text{dom}_{\mathcal{A}}(r) = \bigcap_{j=1}^k \text{dom}_{\mathcal{A}}(Q_j)$  that  $(X_1, \dots, X_g) \in \text{dom}_{\mathcal{A}}(Q^{-1})$  and

$$r(X_1, \dots, X_g) = - \begin{pmatrix} 0 & \dots & 0 & 0 & I_{n_{k+1}} \end{pmatrix} Q(X_1, \dots, X_g)^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ I_{n_1} \end{pmatrix}.$$

In fact, the induction step is based on the observation that if we put

$$\tilde{r} := \Xi_1^{-1} Q_1 \Xi_2^{-1} Q_2 \dots Q_{k-1} \Xi_k^{-1}$$

and correspondingly

$$\tilde{Q} := \begin{pmatrix} & & Q_1 & -\Xi_1 \\ & & \ddots & -\Xi_2 \\ & Q_{k-1} & \ddots & \\ -\Xi_k & & & \end{pmatrix},$$

then we obtain a block decomposition

$$Q = \left( \begin{array}{c|ccc} 0 & & & \\ \vdots & & & \\ 0 & & \tilde{Q} & \\ Q_k & & & \\ \hline -\Xi_k & 0 & \dots & 0 & 0 \end{array} \right).$$

*Example 4.14.* Consider the following slight modification of the rational expression  $r(x_1, x_2)$  that already appeared in Example 3.1, namely

$r(x_1, x_2) = (4-x_1)^{-1} + (4-x_1)^{-1} x_2 ((4-x_1) - x_2(4-x_1)^{-1} x_2)^{-1} x_2 (4-x_1)^{-1}$ , which admits the selfadjoint realization

$$r(x_1, x_2) := \begin{pmatrix} \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 - \frac{1}{4}x_1 & -\frac{1}{4}x_2 \\ -\frac{1}{4}x_2 & 1 - \frac{1}{4}x_1 \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}.$$

According to Theorem 4.7, we introduce

$$\hat{\Lambda}(x_1, x_2) = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 + \frac{1}{4}x_1 & \frac{1}{4}x_2 \\ 0 & \frac{1}{4}x_2 & -1 + \frac{1}{4}x_1 \end{pmatrix},$$



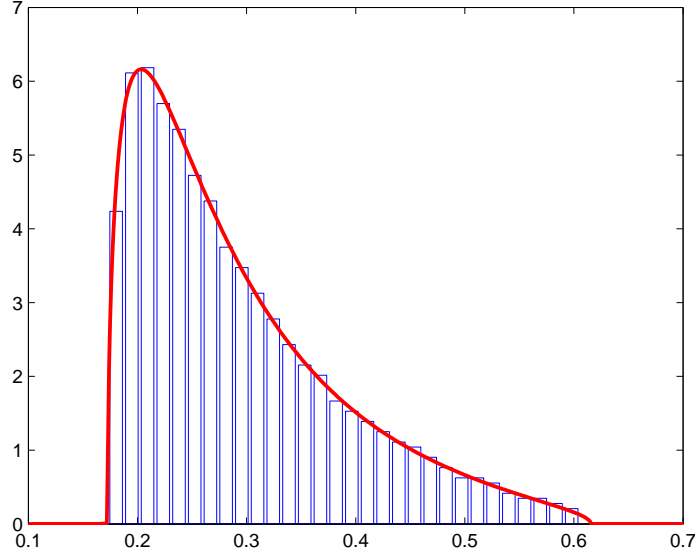


FIGURE 3. Histogram of eigenvalues of  $r(X_1^{(N)}, X_2^{(N)})$  for one realization of independent Gaussian random matrices  $X_1^{(N)}, X_2^{(N)}$  of size  $N = 1000$ , compared with the distribution of  $r(X_1, X_2)$  for freely independent semicircular elements  $X_1, X_2$ . See Example 4.14.

which decomposes as  $\hat{\Lambda}(x_1, x_2) = \hat{\Lambda}_0 + \hat{\Lambda}_1 x_1 + \hat{\Lambda}_2 x_2$ , where

$$\hat{\Lambda}_0 = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \hat{\Lambda}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}, \quad \text{and} \quad \hat{\Lambda}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 \end{pmatrix}.$$

In Figure 3, we compare the histogram of eigenvalues of  $r(X_1^{(N)}, X_2^{(N)})$  for one realization of independent Gaussian random matrices  $X_1^{(N)}, X_2^{(N)}$  of size  $N = 1000$  with the distribution of  $r(X_1, X_2)$  for freely independent semicircular elements  $X_1, X_2$ , calculated according to our algorithm.  $\square$

*Example 4.15.* We consider now the rational expression  $r(x_1, x_2)$ , determined by its realization

$$r(x_1, x_2) := \begin{pmatrix} 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 - \frac{1}{4}x_1 & -ix_2 \\ -\frac{1}{4}x_2 & 1 - \frac{1}{4}x_1 \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}.$$

Using the construction that we presented in Lemma 4.10, we obtain a realization of

$$\mathbb{r}(x) = \begin{pmatrix} 0 & r(x) \\ r^*(x) & 0 \end{pmatrix}$$

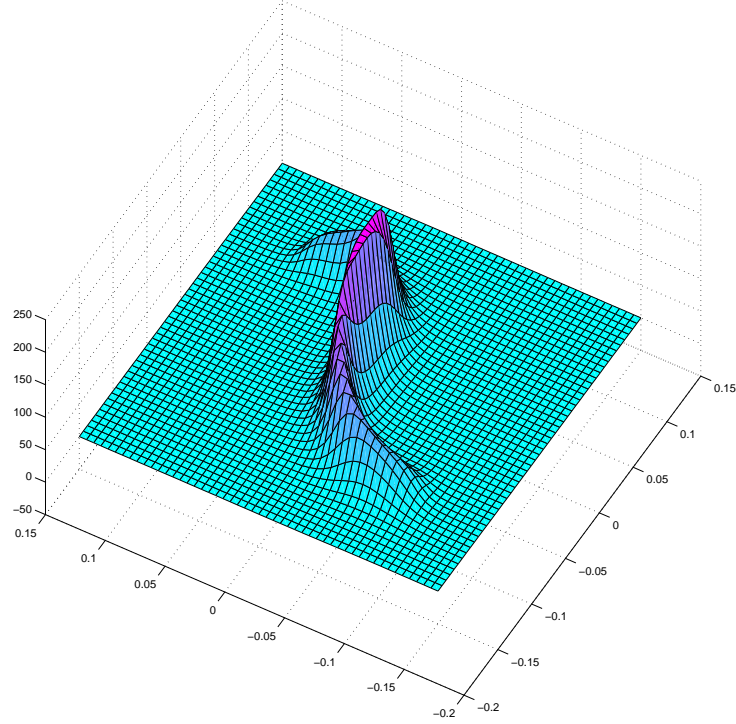


FIGURE 4. Brown measure of  $r(X_1, X_2)$  for the rational expression  $r(x_1, x_2)$  defined in Example 4.15, evaluated in freely independent semicircular elements  $X_1, X_2$ .

by

$$\mathbb{r}(x_1, x_2) = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 - \frac{1}{4}x_1 & -ix_2 \\ 0 & 0 & -\frac{1}{4}x_2 & 1 - \frac{1}{4}x_1 \\ 1 - \frac{1}{4}x_1 & -\frac{1}{4}x_2 & 0 & 0 \\ ix_2 & 1 - \frac{1}{4}x_1 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \\ 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix}.$$

According to Theorem 4.7, we introduce now

$$\hat{\Lambda}(x_1, x_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & -1 + \frac{1}{4}x_1 & ix_2 \\ 0 & 0 & 0 & 0 & \frac{1}{4}x_2 & -1 + \frac{1}{4}x_1 \\ 0 & 0 & -1 + \frac{1}{4}x_1 & \frac{1}{4}x_2 & 0 & 0 \\ \frac{1}{2} & 0 & -ix_2 & -1 + \frac{1}{4}x_1 & 0 & 0 \end{pmatrix}.$$

Again,  $\hat{\Lambda}(x_1, x_2)$  decomposes as  $\hat{\Lambda}(x_1, x_2) = \hat{\Lambda}_0 + \hat{\Lambda}_1 x_1 + \hat{\Lambda}_2 x_2$ , which provides the initial data for our algorithm: if  $X_1, X_2$  are freely independent semicircular elements, then the obtained density of the Brown measure of  $r(X_1, X_2)$  is shown in Figure 4, whereas Figure 5 shows the eigenvalues of  $r(X_1^{(N)}, X_2^{(N)})$

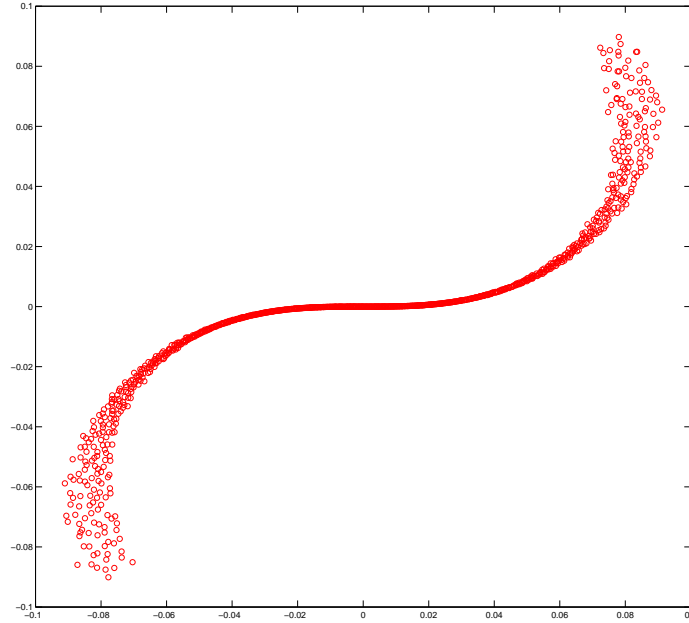


FIGURE 5. Eigenvalues of  $r(X_1^{(N)}, X_2^{(N)})$  for the rational expression  $r(x_1, x_2)$  defined in Example 4.15 with independent Gaussian random matrices  $X_1^{(N)}, X_2^{(N)}$  of size  $N = 1000$ .

for one realization of independent Gaussian random matrices  $X_1^{(N)}, X_2^{(N)}$  of size  $N = 1000$ .  $\square$

## 5. EXPLICIT ALGORITHM FOR A REALIZATION

Here we present an algorithm for constructing a realization of a rational expression  $r$  while keeping a close watch on evaluation properties of both  $r$  and its realization.

We prove existence of a not necessarily minimal realization which has excellent domain and evaluation properties with respect to a algebra  $\mathcal{A}$ , with stably finite not needed.

One motivation is that in our free probability applications in Section 4 it is important to know that the  $\mathcal{A}$ -domain of the considered realization is not smaller than the  $\mathcal{A}$ -domain of the rational expression we are interested in. Hence we need some control over this. By [KVV09] it follows that in the case  $\mathcal{A} = \mathbb{M}(\mathbb{C})$  the minimal realization has the largest possible domain. We will show the validity of this for more general  $\mathcal{A}$  by presenting a concrete realization algorithm.

The algorithmic construction of realizations we present here is not restricted to the regular case, but works for any rational expression. This means that

this algorithm will also apply to the general situation of the full free field, which we are planning to address in a subsequent paper.

Indeed our construction is closely related to similar considerations in the context of the universal skew field of non-commutative rational functions [Co71, Mal78, Co06]. (In the context of regular expressions this goes in principle back to the work of Kleene and Schützenberger, c.f. [K56, S61, S65]. Also an algorithm for the regular case, a bit less general than here, appears in [Stthesis] Chapter 5, and is implemented in *NCAAlgebra* a noncommutative algebra package which runs under Mathematica.

Note that in this paper we have not presented a rigorous definition of the free field of non-commutative rational functions. We merely introduced (rigorously) the subalgebra of noncommutative rational expressions (their equivalence classes being rational functions) regular (at 0). This is a very large subalgebra of the free field, so the machinery we have developed should suffice of most purposes.

In this section we do not produce a minimal realization with good evaluation properties. This requires “cutting down”.

The main expedience of assuming regularity is that the cutting down arguments we saw in Sections 3.2.1 and 3.2.2 behave well. Without regularity complications arise. This case is treated in [Vol15].

Nevertheless, arguments not involving cutdowns work well even without assuming regularity and they can be treated without using results of [Vol15].

Also in the context of this section we prove that the assumption of stable finiteness of the algebra  $\mathcal{A}$  is not needed to build a realization. This is because stably finite is only relevant if one is moving algebraically between different rational expressions of the same rational function and here we keep track of domains and evaluations through each such step to show they are valid for any  $\mathcal{A}$ .

The notation for realizations here is distinct from that used in other parts of this paper. This marks the transition from the present regular context to the more algebraic considerations without regularity assumptions.

**Definition 5.1.** Let  $r$  be a rational expression in the formal variables  $x = (x_1, \dots, x_g)$ . A **formal linear representation**  $\rho = (u, Q, v)$  of  $r$  consists of

- an affine linear pencil

$$Q = Q^{(0)} + Q^{(1)}x_1 + \dots + Q^{(g)}x_g$$

for matrices  $Q^{(0)}, Q^{(1)}, \dots, Q^{(g)} \in M_n(\mathbb{C})$  of some dimension  $n$ ,

- a  $1 \times n$ -matrix  $u$  over  $\mathbb{C}$ ,
- and a  $n \times 1$ -matrix  $v$  over  $\mathbb{C}$

and it satisfies the following property:

For any unital complex algebra  $\mathcal{A}$ , we have that

$$\text{dom}_{\mathcal{A}}(r) \subseteq \text{dom}_{\mathcal{A}}(Q^{-1})$$

and it holds true for any  $(X_1, \dots, X_g) \in \text{dom}_{\mathcal{A}}(r)$  that

$$r(X_1, \dots, X_g) = -uQ(X_1, \dots, X_g)^{-1}v.$$

□

The main contribution of this section is the following algorithm by which we ensure that a formal linear representation exists for any rational expression. Note that the definition of a formal linear representation requires that the domain of definition of the representation includes the domain of definition of the rational expression.

**Theorem 5.2.** *For any rational expression (not necessarily regular at zero) there exists a formal linear representation in the sense of Definition 5.1.*

Later, we will also address a selfadjoint version and an operator-valued generalization of this result; see, in particular, Theorems 5.9 and 5.14.

The proof of Theorem 5.2 is provided by the following algorithm for producing such a formal linear representation. Recall that any rational expression is build from scalars  $\lambda \in \mathbb{C}$  and the formal variables  $x_1, \dots, x_g$  by applying iteratively the operations  $+$ ,  $\cdot$ , and  $\cdot^{-1}$ .

*Algorithm 5.3.* Let  $r$  be a rational expression in the formal variables  $x = (x_1, \dots, x_g)$ . A formal linear representation  $\rho = (u, Q, v)$  of  $r$  can be constructed by using successively the following rules:

- (i) For scalars  $\lambda \in \mathbb{C}$  and the variables  $x_j$ ,  $j = 1, \dots, g$ , formal linear representations are given by

$$(5.1) \quad \begin{aligned} \rho_{x_j} &:= \left( (0 \ 1), \begin{pmatrix} x_j & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \quad \text{and} \\ \rho_\lambda &:= \left( (0 \ 1), \begin{pmatrix} \lambda & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right), \end{aligned}$$

respectively.

- (ii) If  $\rho_1 = (u_1, Q_1, v_1)$  and  $\rho_2 = (u_2, Q_2, v_2)$  are formal linear representations for the rational expressions  $r_1$  and  $r_2$ , respectively, then

$$(5.2) \quad \rho_1 \oplus \rho_2 := \left( (u_1 \ u_2), \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right)$$

gives a formal linear representation of  $r_1 + r_2$ .

- (iii) If  $\rho_1 = (u_1, Q_1, v_1)$  and  $\rho_2 = (u_2, Q_2, v_2)$  are formal linear representations for the rational expressions  $r_1$  and  $r_2$ , respectively, then

$$(5.3) \quad \rho_1 \odot \rho_2 := \left( (0 \ u_1), \begin{pmatrix} v_1 u_2 & Q_1 \\ Q_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \right)$$

gives a formal linear representation of  $r_1 \cdot r_2$ .

- (iv) If  $\rho = (u, Q, v)$  is a formal linear representation of  $r$ , then

$$(5.4) \quad \rho^{-1} := \left( (1 \ 0), \begin{pmatrix} 0 & u \\ v & -Q \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$

gives a formal linear representation of  $r^{-1}$ .

Note that the operations (5.1), (5.2), (5.3), and (5.4), which we described in Algorithm 5.3, have to be understood on the level of linear pencils. More precisely, we use here the natural convention that a matrix

$$Q = \begin{pmatrix} Q_{1,1} & \cdots & Q_{1,m} \\ \vdots & \ddots & \vdots \\ Q_{m,1} & \cdots & Q_{m,m} \end{pmatrix}$$

of linear pencils

$$Q_{k,l} = Q_{k,l}^{(0)} + Q_{k,l}^{(1)}x_1 + \cdots + Q_{k,l}^{(g)}x_g, \quad 1 \leq k, l \leq m,$$

with matrices  $Q_{k,l}^{(0)}, Q_{k,l}^{(1)}, \dots, Q_{k,l}^{(g)} \in M_{n_k \times n_l}(\mathbb{C})$  of certain dimensions  $n_1, \dots, n_m$ , represents itself a linear pencil

$$Q = Q^{(0)} + Q^{(1)}x_1 + \cdots + Q^{(g)}x_g,$$

where we put

$$Q^{(j)} := \begin{pmatrix} Q_{1,1}^{(j)} & \cdots & Q_{1,m}^{(j)} \\ \vdots & \ddots & \vdots \\ Q_{m,1}^{(j)} & \cdots & Q_{m,m}^{(j)} \end{pmatrix}, \quad \text{for } j = 1, \dots, g.$$

Furthermore, if a linear pencil

$$Q = Q^{(0)} + Q^{(1)}x_1 + \cdots + Q^{(g)}x_g$$

of dimension  $n$  and matrices  $S, T \in M_n(\mathbb{C})$  are given, we will denote by  $SQT$  the linear pencil that is defined by

$$SQT := SQ^{(0)}T + (SQ^{(1)}T)x_1 + \cdots + (SQ^{(g)}T)x_g.$$

The proof that the rules (i) – (iv) given in the above algorithm are indeed correct, will be given in Subsection 5.1.

*Example 5.4.* We consider the rational expressions

$$r_1 = (x_1x_2)^{-1} \quad \text{and} \quad r_2 = x_2^{-1}x_1^{-1}$$

By applying Algorithm 5.3, we obtain for  $r_1$  the formal linear representation

$$\rho_1 = \left( (1 \ 0 \ 0 \ 0 \ 0), \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -x_1 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -x_2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right)$$

and for  $r_2$  the formal linear representation

$$\rho_2 = \left( (0 \ 0 \ 0 \ 1 \ 0 \ 0), \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & x_2 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & x_1 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right).$$

This highlights the computational disadvantage of Algorithm 5.3, that roughly speaking the dimension of the linear pencil  $Q$  of a formal linear representation  $\rho = (u, Q, v)$  increases rapidly with the complexity of the rational expression  $r$  that it represents. Clearly, since the rational expressions  $r_1$  and  $r_2$  in the example above are rather simple, we would expect that there are other formal linear representations of smaller dimensions. Unfortunately, since  $r_1$  and  $r_2$  are both not regular, we cannot use the representation machinery to cut down our realizations to minimal ones. One expedient could be to use the analogous but more general machinery that was invented recently in [Vol15]. However, there are also some ad hoc constructions: because any formal linear representation  $\rho = (u, Q, v)$  of a rational expression  $r$  can clearly be transformed by

$$S \cdot \rho \cdot T := (uT, SQT, Sv),$$

for any choice of invertible matrices  $S, T \in M_n(\mathbb{C})$ , to another formal linear representation of  $r$ , there is at least an expedient: if we arrange  $\rho = (u, Q, v)$  as

$$\begin{array}{c|c} & u \\ \hline v & Q \end{array},$$

we can try to bring this array into the form

$$\begin{array}{c|cc} & \tilde{u} & u' \\ \hline \tilde{v} & Q & 0 \\ v' & 0 & Q' \end{array},$$

by acting by elementary row and column operations only on  $Q$ , while book-keeping their effect in the first row and column, respectively. If it happens in this case that  $(u', Q', v')$  is a formal linear representation of 0, we can just remove this part, which means that  $\tilde{\rho} = (\tilde{u}, \tilde{Q}, \tilde{v})$  gives another formal linear representation of  $r$ .

In fact, we can show by using this method that

$$\tilde{\rho}_1 = (\tilde{u}_1, \tilde{Q}_1, \tilde{v}_1) = \left( (1 \ 0), \begin{pmatrix} 0 & x_1 \\ x_2 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$

gives another formal linear representation of  $r_1$  and that

$$\tilde{\rho}_2 = (\tilde{u}_2, \tilde{Q}_2, \tilde{v}_2) = \left( (0 \ 1), \begin{pmatrix} 1 & -x_2 \\ -x_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

gives another formal linear representation of  $r_2$ .

It is easy to see that the linear pencils  $Q_1, Q_2$  satisfy the relation

$$Q_1 = UQ_2U^{-1} \quad \text{where} \quad U := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Thus, we have  $\text{dom}_{\mathcal{A}}(Q_1^{-1}) = \text{dom}_{\mathcal{A}}(Q_2^{-1})$  for any unital complex algebra  $\mathcal{A}$ .

Furthermore, we obtain by using the Schur complement formula, that  $Q_1(X_1, X_2)$  (and hence  $Q_2(X_1, X_2)$ ) is invertible in  $M_2(\mathcal{A})$  for some tuple  $(X_1, X_2)$  of elements in a unital complex algebra  $\mathcal{A}$ , if and only if  $X_1X_2$  is invertible in  $\mathcal{A}$ .



In other words, we have

$$\text{dom}_{\mathcal{A}}(r_2) \subsetneq \text{dom}_{\mathcal{A}}(r_1) = \text{dom}_{\mathcal{A}}(Q_1^{-1}) = \text{dom}_{\mathcal{A}}(Q_2^{-1}).$$

**5.1. Proof of Rules in Algorithm 5.3.** First of all, we examine the validity of rule (i). The following lemma gives a slightly more general statement and allows a uniform proof for formal linear representations of scalars  $\lambda \in \mathbb{C}$  and the variables  $x_1, \dots, x_g$ .

**Lemma 5.5.** *Consider a rational expression  $r$  of the form*

$$r = \lambda_0 + \lambda_1 x_1 + \dots + \lambda_g x_g$$

*for some  $\lambda_0, \lambda_1, \dots, \lambda_g \in \mathbb{C}$ . Then a formal linear representation of  $r$  is given by*

$$\rho := \left( (0 \ 1), \begin{pmatrix} r & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right).$$

*Proof.* Write  $\rho = (u, Q, v)$ . First of all, we note that

$$Q = \begin{pmatrix} \lambda_0 & -1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} \lambda_1 & 0 \\ 0 & 0 \end{pmatrix} x_1 + \dots + \begin{pmatrix} \lambda_g & 0 \\ 0 & 0 \end{pmatrix} x_g.$$

Now, consider any unital complex algebra  $\mathcal{A}$ . We observe that the matrix  $Q(X)$  is invertible for any  $X = (X_1, \dots, X_g) \in \text{dom}_{\mathcal{A}}(r) = \mathcal{A}^g$  with

$$Q(X)^{-1} = \begin{pmatrix} 0 & -1 \\ -1 & -r(X) \end{pmatrix}.$$

Hence  $X \in \text{dom}_{\mathcal{A}}(Q^{-1})$  and furthermore  $-uQ(X)^{-1}v = r(X)$ , which completes the proof that  $\rho$  is a formal linear representation of  $r$  in the sense of Definition 5.1.  $\square$

Next, we give a lemma that justifies the rules (ii) and (iii).

**Lemma 5.6.** *Let  $\rho_1 = (u_1, Q_1, v_1)$  and  $\rho_2 = (u_2, Q_2, v_2)$  be formal linear representations of rational expressions  $r_1$  and  $r_2$ , respectively. Then the following statements hold true:*

- *A formal linear representation of  $r_1 + r_2$  is given by*

$$\rho_1 \oplus \rho_2 := \left( (u_1 \ u_2), \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right).$$

- *A formal linear representation of  $r_1 \cdot r_2$  is given by*

$$\rho_1 \odot \rho_2 := \left( (0 \ u_1), \begin{pmatrix} v_1 u_2 & Q_1 \\ Q_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \right).$$

*Proof.* For any unital complex algebra  $\mathcal{A}$ , consider  $X = (X_1, \dots, X_g) \in \text{dom}_{\mathcal{A}}(r_1 + r_2) = \text{dom}_{\mathcal{A}}(r_1) \cap \text{dom}_{\mathcal{A}}(r_2)$ . Since  $\rho_1$  and  $\rho_2$  are both formal linear representations, we have

$$r_1(X) = -u_1 Q_1(X)^{-1} v_1 \quad \text{and} \quad r_2(X) = -u_2 Q_2(X)^{-1} v_2.$$

For  $\rho_1 \oplus \rho_2 = (u, Q, v)$ , this means in particular that the matrix

$$Q(X) = \begin{pmatrix} Q_1(X) & 0 \\ 0 & Q_2(X) \end{pmatrix}$$

is invertible, which shows  $X \in \text{dom}_{\mathcal{A}}(Q^{-1})$ , and moreover allows us to check

$$\begin{aligned} -uQ(X)^{-1}v &= -\begin{pmatrix} u_1 & u_2 \end{pmatrix} \begin{pmatrix} Q_1(X)^{-1} & 0 \\ 0 & Q_2(X)^{-1} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ &= -u_1Q_1(X)^{-1}v_1 - u_2Q_2(X)^{-1}v_2 \\ &= r_1(X) + r_2(X). \end{aligned}$$

Since  $X \in \text{dom}_{\mathcal{A}}(r_1 + r_2)$  was arbitrarily chosen, we conclude that  $\rho_1 \oplus \rho_2$  is a formal linear representation of  $r_1 + r_2$ .

Similarly, if we consider  $X \in \text{dom}_{\mathcal{A}}(r_1 \cdot r_2) = \text{dom}_{\mathcal{A}}(r_1) \cap \text{dom}_{\mathcal{A}}(r_2)$ , we obtain for  $\rho_1 \odot \rho_2 = (u, Q, v)$  the invertibility of the matrix

$$Q(X) = \begin{pmatrix} v_1u_2 & Q_1(X) \\ Q_2(X) & 0 \end{pmatrix}.$$

In fact, one can convince oneself by a straightforward computation that more precisely

$$Q(X)^{-1} = \begin{pmatrix} 0 & Q_2(X)^{-1} \\ Q_1(X)^{-1} & -Q_1(X)^{-1}v_1u_2Q_2(X)^{-1} \end{pmatrix}.$$

This proves  $X \in \text{dom}_{\mathcal{A}}(Q^{-1})$  and allows us to check

$$\begin{aligned} -uQ(X)^{-1}v &= -\begin{pmatrix} 0 & u_1 \end{pmatrix} \begin{pmatrix} 0 & Q_2(X)^{-1} \\ Q_1(X)^{-1} & -Q_1(X)^{-1}v_1u_2Q_2(X)^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \\ &= u_1Q_1(X)^{-1}v_1u_2Q_2(X)^{-1}v_2 \\ &= r_1(X)r_2(X). \end{aligned}$$

Since  $X \in \text{dom}_{\mathcal{A}}(r_1 \cdot r_2)$  was again arbitrarily chosen, we may conclude now that  $\rho_1 \odot \rho_2$  gives as stated a formal linear representation of  $r_1 \cdot r_2$ .  $\square$

Finally, concerning rule (iv) of Algorithm 5.3, we show the following lemma.

**Lemma 5.7.** *Let  $\rho = (u, Q, v)$  be a formal linear representation of a rational expression  $r$ . Then*

$$\rho^{-1} := \left( \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & u \\ v & -Q \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$

*gives a formal linear representation of  $r^{-1}$ .*

*Proof.* Take any  $X \in \text{dom}_{\mathcal{A}}(r^{-1})$ , which means by definition that  $X \in \text{dom}_{\mathcal{A}}(r)$  and that  $r(X) \in \mathcal{A}$  is invertible. Since  $\rho$  is assumed to be a formal linear representation of  $r$ , this ensures the invertibility of  $Q(X)$ . Hence, the Schur complement formula tells us that the matrix

$$\begin{pmatrix} 0 & u \\ v & -Q(X) \end{pmatrix}$$

must be invertible since its Schur complement is given by  $uQ(X)^{-1}v = -r(X)$ . Hence, we infer  $X \in \text{dom}_{\mathcal{A}} \left( \begin{pmatrix} 0 & u \\ v & -Q \end{pmatrix}^{-1} \right)$ . Furthermore, the Schur complement formula tells us in this case that

$$(1 \ 0) \begin{pmatrix} 0 & u \\ v & -Q(X) \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -(uQ(X)^{-1}v)^{-1} = r(X).$$

Since this holds for all  $X \in \text{dom}_{\mathcal{A}}(r^{-1})$ , we see that  $\rho^{-1}$  is indeed a formal linear representation of  $r^{-1}$ .  $\square$

**5.2. selfadjoint formal linear representations.** Let  $r$  be any rational expression in the formal variables  $x = (x_1, \dots, x_g)$ . If  $\mathcal{A}$  is a unital complex  $*$ -algebra, we denote by  $\text{dom}_{\mathcal{A}}^{\text{sa}}(r)$  the subset of  $\text{dom}_{\mathcal{A}}(r)$  that consists of all selfadjoint points  $X = (X_1, \dots, X_g) \in \text{dom}_{\mathcal{A}}(r)$ , i.e. we have  $X = X^*$ , where  $X^* := (X_1^*, \dots, X_g^*)$ .

We say that the rational expression  $r$  is **selfadjoint**, if for any unital complex  $*$ -algebra  $\mathcal{A}$  and for any  $X \in \text{dom}_{\mathcal{A}}^{\text{sa}}(r)$  it holds true that  $r(X)^* = r(X)$ . Note that this definition slightly differs from the usual terminology of symmetric rational expressions in the real case.

**Definition 5.8.** Let  $r$  be a selfadjoint rational expression. A **selfadjoint formal linear representation**  $\rho = (Q, v)$  consists of

- an affine linear pencil

$$Q = Q^{(0)} + Q^{(1)}x_1 + \dots + Q^{(g)}x_g$$

for selfadjoint matrices  $Q^{(0)}, Q^{(1)}, \dots, Q^{(g)} \in M_n(\mathbb{C})$  of some dimension  $n$

- and a  $n \times 1$ -matrix  $v$  over  $\mathbb{C}$

and it satisfies the following property:

For any unital complex  $*$ -algebra  $\mathcal{A}$ , we have that

$$\text{dom}_{\mathcal{A}}^{\text{sa}}(r) \subseteq \text{dom}_{\mathcal{A}}(Q^{-1})$$

and it holds true for any  $(X_1, \dots, X_g) \in \text{dom}_{\mathcal{A}}^{\text{sa}}(r)$  that

$$r(X_1, \dots, X_g) = -v^*Q(X_1, \dots, X_g)^{-1}v.$$

$\square$

**Theorem 5.9.** Any selfadjoint rational expression  $r$  (not necessarily regular at zero) admits a selfadjoint formal linear representation  $\rho = (Q, v)$  in the sense of Definition 5.8.

*Proof.* We consider any formal linear representation  $\rho_0 = (u_0, Q_0, v_0)$  of  $r$  and we put

$$(5.5) \quad \rho = (Q, v) := \left( \begin{pmatrix} 0 & Q_0^* \\ Q_0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2}u_0^* \\ v_0 \end{pmatrix} \right).$$

Clearly, the linear pencil  $Q$  consists of selfadjoint matrices and satisfies  $\text{dom}_{\mathcal{A}}(Q_0^{-1}) = \text{dom}_{\mathcal{A}}(Q^{-1})$  for any unital complex  $(*)$ -algebra  $\mathcal{A}$ , since we have for arbitrary

$X = (X_1, \dots, X_g) \in \mathcal{A}^g$  that  $Q(X)$  is invertible if and only if  $Q_0(X)$  is invertible.

Furthermore, if  $\mathcal{A}$  is any unital complex  $*$ -algebra, we have

$$\text{dom}_{\mathcal{A}}^{\text{sa}}(r) \subseteq \text{dom}_{\mathcal{A}}(r) \subseteq \text{dom}_{\mathcal{A}}(Q_0^{-1}) = \text{dom}_{\mathcal{A}}(Q^{-1})$$

and we may observe that for each point  $X \in \text{dom}_{\mathcal{A}}^{\text{sa}}(r)$

$$\begin{aligned} -uQ(X)^{-1}v &= \begin{pmatrix} \frac{1}{2}u_0 & v_0^* \end{pmatrix} \begin{pmatrix} 0 & Q_0(X)^* \\ (Q_0(X)^*)^{-1} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2}u_0^* \\ v_0 \end{pmatrix} \\ &= -\frac{1}{2}u_0Q_0(X)^{-1}v_0 - \frac{1}{2}v_0^*(Q_0(X)^*)^{-1}u_0^* \\ &= -\frac{1}{2}u_0Q_0(X)^{-1}v_0 - \frac{1}{2}(u_0Q_0(X)^{-1}v_0)^* \\ &= \frac{1}{2}r(X) + \frac{1}{2}r(X)^* \\ &= r(X). \end{aligned}$$

This completes the proof.  $\square$

Note that we used in Theorem 4.7 that each selfadjoint formal linear representation  $\rho = (Q, v)$  of a selfadjoint rational expression  $r$  induces a selfadjoint linearization  $\hat{\Lambda}$  by

$$\hat{\Lambda} := \begin{pmatrix} 0 & v^* \\ v & Q \end{pmatrix}.$$

### 5.3. Operator-valued rational expressions and their representations.

We conclude by noting that the theory presented above can be extended to matrices of rational expressions. In fact, we may extend it more generally to the operator-valued case, which arises, roughly speaking, if the role of the complex numbers  $\mathbb{C}$  is taken over by any other complex unital algebra  $\mathcal{B}$ . In the case where this algebra  $\mathcal{B}$  is given by  $M_n(\mathbb{C})$ , we cover in particular the important case of matrices of rational expressions.

More formally, a  **$\mathcal{B}$ -valued rational expression** in the formal variables  $x = (x_1, \dots, x_g)$  is built from all elements in  $\mathcal{B}$  and the formal variables  $x_1, \dots, x_g$  by applying successively the operations  $+$ ,  $\cdot$ , and  $\cdot^{-1}$ .

For any unital complex algebra  $\mathcal{A}$  which contains  $\mathcal{B}$  as a subalgebra with identified units, we define the  $\mathcal{A}$ -domain  $\text{dom}_{\mathcal{A}:\mathcal{B}}(r)$  of  $\mathcal{A}$  in the same way as in the scalar-valued case  $\mathcal{B} = \mathbb{C}$ .

As a matter of fact, since a linear pencil  $Q$  of matrices of dimension  $n$  is in particular a  $M_n(\mathbb{C})$ -valued rational expression, we have the compatibility condition  $\text{dom}_{\mathcal{A}}(Q^{-1}) = \mathcal{A}^g \cap \text{dom}_{M_n(\mathcal{A}):M_n(\mathbb{C})}(Q^{-1})$ , where  $\mathcal{A}$  is seen as a subalgebra of  $M_n(\mathcal{A}) \cong M_n(\mathbb{C}) \otimes \mathcal{A}$  by  $X \mapsto 1 \otimes X$ .

**Definition 5.10.** Let  $r$  be a  $\mathcal{B}$ -valued rational expression in the formal variables  $x = (x_1, \dots, x_g)$ . A  **$\mathcal{B}$ -valued formal linear representation**  $\rho = (u, Q, v)$  of  $r$  consists of

- an affine linear pencil

$$Q = Q^{(0)} + Q^{(1)}x_1 + \cdots + Q^{(g)}x_g$$

for matrices  $Q^{(0)}, Q^{(1)}, \dots, Q^{(g)} \in M_n(\mathcal{B})$  of some dimension  $n$ ,

- a  $1 \times n$ -matrix  $u$  over  $\mathcal{B}$ ,
- and a  $n \times 1$ -matrix  $v$  over  $\mathcal{B}$

and it satisfies the following property:

For any unital complex algebra  $\mathcal{A}$  with  $1 \in \mathcal{B} \subseteq \mathcal{A}$ , we have that

$$\text{dom}_{\mathcal{A}:\mathcal{B}}(r) \subseteq \text{dom}_{\mathcal{A}:\mathcal{B}}(Q^{-1})$$

and it holds true for any  $(X_1, \dots, X_g) \in \text{dom}_{\mathcal{A}:\mathcal{B}}(r)$  that

$$r(X_1, \dots, X_g) = -uQ(X_1, \dots, X_g)^{-1}v.$$

□

Clearly, the domain  $\text{dom}_{\mathcal{A}:\mathcal{B}}(Q^{-1})$  is defined by

$$\text{dom}_{\mathcal{A}:\mathcal{B}}(Q^{-1}) := \{(X_1, \dots, X_g) \in \mathcal{A}^g \mid Q(X_1, \dots, X_g) \in M_n(\mathcal{A}) \text{ invertible}\}$$

and it agrees with  $\mathcal{A}^g \cap \text{dom}_{M_n(\mathcal{A}):M_n(\mathcal{B})}(Q^{-1})$ .

It is easy to see that Algorithm 5.3 extends immediately to the case of  $\mathcal{B}$ -valued rational expressions.

*Algorithm 5.11.* Let  $r$  be an  $\mathcal{B}$ -valued rational expression in the formal variables  $x = (x_1, \dots, x_g)$ . A formal linear representation  $\rho = (u, Q, v)$  of  $r$  can be constructed by using successively the following rules:

- (i) For scalars  $b \in \mathcal{B}$  and the variables  $x_j$ ,  $j = 1, \dots, g$ ,  $\mathcal{B}$ -valued formal linear representations are given by

$$\rho_{x_j} := \left( \begin{pmatrix} 0 & 1 \end{pmatrix}, \begin{pmatrix} x_j & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

and

$$\rho_b := \left( \begin{pmatrix} 0 & 1 \end{pmatrix}, \begin{pmatrix} b & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right),$$

respectively.

- (ii) If  $\rho_1 = (u_1, Q_1, v_1)$  and  $\rho_2 = (u_2, Q_2, v_2)$  are  $\mathcal{B}$ -valued formal linear representations for the  $\mathcal{B}$ -valued rational expressions  $r_1$  and  $r_2$ , respectively, then  $\rho_1 \oplus \rho_2$  as defined in (5.2) gives a  $\mathcal{B}$ -valued formal linear representation of  $r_1 + r_2$ .
- (iii) If  $\rho_1 = (u_1, Q_1, v_1)$  and  $\rho_2 = (u_2, Q_2, v_2)$  are  $\mathcal{B}$ -valued formal linear representations for the  $\mathcal{B}$ -valued rational expressions  $r_1$  and  $r_2$ , respectively, then  $\rho_1 \odot \rho_2$  as defined in (5.3) gives a  $\mathcal{B}$ -valued formal linear representation of  $r_1 \cdot r_2$ .
- (iv) If  $\rho = (u, Q, v)$  is a  $\mathcal{B}$ -valued formal linear representation of  $r$ , then  $\rho^{-1}$  as defined in (5.4) gives a  $\mathcal{B}$ -valued formal linear representation of  $r^{-1}$ .

Thus, we obtain the following operator-valued analogue of Theorem 5.2.

**Theorem 5.12.** *Let  $\mathcal{B}$  be a unital complex algebra. Each  $\mathcal{B}$ -valued rational expression has an  $\mathcal{B}$ -valued formal linear representation in the sense of Definition 5.10.*

If  $\mathcal{B}$  is moreover a  $*$ -algebra, we may introduce like in the scalar-valued case the notion of selfadjoint  $\mathcal{B}$ -valued rational expressions. More precisely, we will call a  $\mathcal{B}$ -valued rational expression  $r$  **selfadjoint**, if for any unital complex  $*$ -algebra  $\mathcal{A}$  that contains  $\mathcal{B}$  as a  $*$ -subalgebra and for any  $X \in \text{dom}_{\mathcal{A}:\mathcal{B}}^{\text{sa}}(r)$  it holds true that  $r(X)^* = r(X)$ .

Also Definition 5.8 extends to this generality.

**Definition 5.13.** Let  $r$  be a selfadjoint  $\mathcal{B}$ -valued rational expression. A **self-adjoint  $\mathcal{B}$ -valued formal linear representation**  $\rho = (Q, v)$  consists of

- an affine linear pencil

$$Q = Q^{(0)} + Q^{(1)}x_1 + \cdots + Q^{(g)}x_g$$

for selfadjoint matrices  $Q^{(0)}, Q^{(1)}, \dots, Q^{(g)} \in M_n(\mathcal{B})$  of some dimension  $n$

- and a  $n \times 1$ -matrix  $v$  over  $\mathcal{B}$

and it satisfies the following property:

For any unital complex  $*$ -algebra  $\mathcal{A}$  that contains  $\mathcal{B}$  as a  $*$ -subalgebra, we have that

$$\text{dom}_{\mathcal{A}:\mathcal{B}}^{\text{sa}}(r) \subseteq \text{dom}_{\mathcal{A}:\mathcal{B}}(Q^{-1})$$

and it holds true for any  $(X_1, \dots, X_g) \in \text{dom}_{\mathcal{A}:\mathcal{B}}^{\text{sa}}(r)$  that

$$r(X_1, \dots, X_g) = -v^*Q(X_1, \dots, X_g)^{-1}v.$$

□

Consequently, we also have an  $\mathcal{B}$ -valued analogue of Theorem 5.9.

**Theorem 5.14.** *Any selfadjoint  $\mathcal{B}$ -valued rational expression admits a self-adjoint  $\mathcal{B}$ -valued formal linear representation in the sense of Definition 5.13.*

The proof uses exactly the same construction. Namely, for an  $\mathcal{B}$ -valued formal linear representation  $\rho = (u, Q, v)$ , we put

$$\rho = (Q, v) := \left( \begin{pmatrix} 0 & Q_0^* \\ Q_0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2}u_0^* \\ v_0 \end{pmatrix} \right).$$

**5.4. The regular case.** Finally, we turn our attention to the case of (matrices of) rational expressions, which are regular at zero. Besides the classical theory of descriptor realizations, there is now, provided by formal linear representations as discussed in the previous subsections, another concept in the spirit of linearization. Our aim is to clarify the relations between these two approaches. As it will turn out, the excellent evaluation properties of formal linear representations with respect to stably finite algebras pass on realizations under the assumption of minimality.

**Theorem 5.15.** *Let  $r$  be a matrix of rational expressions in formal variables  $x_1, \dots, x_g$ , which is regular at zero. Then the following statements hold true:*

(i) *It admits a monic realization of the form*

$$\mathfrak{r}(x) = D + C(I - L_A(x))^{-1}B,$$

*where the feed through term  $D \in M_k(\mathbb{C})$  can be prescribed arbitrarily, which enjoys the following property:*

*If  $\mathcal{A}$  is a unital complex algebra (not necessarily stably finite), then*

$$\text{dom}_{\mathcal{A}}(r) \subseteq \text{dom}_{\mathcal{A}}(\mathfrak{r})$$

*and*

$$r(X) = \mathfrak{r}(X) \quad \text{if } X \in \text{dom}_{\mathcal{A}}(r).$$

(ii) *Any minimal realization*

$$\hat{\mathfrak{r}}(x) = D + \hat{C}(\hat{J} - L_{\hat{A}}(x))^{-1}\hat{B},$$

*of  $r$  satisfies the following property:*

*If  $\mathcal{A}$  is a unital complex algebra, which is stably finite, then*

$$(5.6) \quad \text{dom}_{\mathcal{A}}(r) \subseteq \text{dom}_{\mathcal{A}}(\hat{\mathfrak{r}})$$

*and*

$$(5.7) \quad r(X) = \hat{\mathfrak{r}}(X) \quad \text{if } X \in \text{dom}_{\mathcal{A}}(r).$$

*Proof.* For proving (i), we proceed as follows: By Theorem 5.12 we may find any matrix-valued formal linear representation  $\rho = (u, Q, v)$  of  $r - D$ . Since  $0 \in \text{dom}_{\mathcal{A}}(r) = \text{dom}_{\mathcal{A}}(r - D)$  holds by the regularity assumption and since we have  $\text{dom}_{\mathcal{A}}(r - D) \subseteq \text{dom}_{\mathcal{A}}(Q^{-1})$  due to Definition 5.10, we see that the linear pencil  $Q$  entails an invertible matrix  $Q^{(0)}$ . Thus, we may introduce

$$\mathfrak{r}_0(x) := -u(I + (Q^{(0)})^{-1}Q^{(1)}x_1 + \dots + (Q^{(0)})^{-1}Q^{(g)}x_g)^{-1}(Q^{(0)})^{-1}v,$$

which is of the form  $C(I - L_A(x))^{-1}B$  with  $C = -u$ ,  $B = (Q^{(0)})^{-1}v$  and  $A_j = -(Q^{(0)})^{-1}Q^{(j)}$  for  $j = 1, \dots, n$ . Again by Definition 5.10, we know that  $\text{dom}_{\mathcal{A}}(r - D) \subseteq \text{dom}_{\mathcal{A}}(\mathfrak{r})$  holds for any unital complex algebra  $\mathcal{A}$  and in addition

$$r(X) - D = \mathfrak{r}_0(X) \quad \text{for all } X \in \text{dom}_{\mathcal{A}}(r),$$

i.e.

$$r(X) = D + C(I - L_A(x))^{-1}B \quad \text{for all } X \in \text{dom}_{\mathcal{A}}(r).$$

Since this applies in particular to the case  $\mathcal{A} = M_N(\mathbb{C})$ , we see that  $r$  and  $\mathfrak{r}$  are equivalent under matrix evaluation and hence power series equivalent, which means that we have found by  $\mathfrak{r}(x) = D + C(I - L_A(x))^{-1}B$  the desired monic descriptor realization of  $r$ .

For seeing (ii), we start with any descriptor realization  $\mathfrak{r}$  of  $r$  as in part (i). By Item 2 of Lemma 3.9, we know that for stably finite  $\mathcal{A}$  its domain  $\text{dom}_{\mathcal{A}}(\mathfrak{r})$ , which contains  $\text{dom}_{\mathcal{A}}(r)$  by the construction in (i), must itself be a subset of the domain of any minimal realization of  $r$ . Thus, for the given minimal realization  $\hat{\mathfrak{r}}$ , we finally get a chain of inclusions

$$\text{dom}_{\mathcal{A}}(r) \subseteq \text{dom}_{\mathcal{A}}(\mathfrak{r}) \subseteq \text{dom}_{\mathcal{A}}(\hat{\mathfrak{r}}),$$



which proves (5.6). Furthermore, for any point in  $\text{dom}_{\mathcal{A}}(r)$ , we know by (i) and Item 2 of Lemma 3.9 that

$$r(X) = \mathfrak{r}(X) = \hat{\mathfrak{r}}(X)$$

holds for any  $X \in \text{dom}_{\mathcal{A}}(r)$ . This shows the validity of (5.7) and concludes the proof.  $\square$

Similarly, selfadjoint representations allow us to construct selfadjoint realizations. At the first glance, this observations seems to be trivial, but its proof is in fact quite intricate. In particular, it relies crucially on [HMOV06, Proposition A.7], according to which rational expressions are  $\mathbb{M}(\mathbb{R})$ -evaluation equivalent, if and only if they are  $\mathbb{M}(\mathbb{R})_{\text{sa}}$ -evaluation equivalent. This will be done in the following theorem, which can be seen as a selfadjoint counterpart of Theorem 5.15.

**Theorem 5.16.** *Let  $r$  be a selfadjoint matrix of rational expressions in formal variables  $x_1, \dots, x_g$ , which is regular at zero. Then the following statements hold true:*

(i) *It admits a selfadjoint realization of the form*

$$\mathfrak{r}(x) = \Delta + \Xi^*(M_0 - L_M(x))^{-1}\Xi,$$

*where the feed through term  $\Delta \in M_k(\mathbb{C})$  can be prescribed arbitrarily, which enjoys the following property:*

*If  $\mathcal{A}$  is a unital complex  $*$ -algebra (not necessarily stably finite), then*

$$\text{dom}_{\mathcal{A}}^{\text{sa}}(r) \subseteq \text{dom}_{\mathcal{A}}(\mathfrak{r})$$

*and*

$$r(X) = \mathfrak{r}(X) \quad \text{if } X \in \text{dom}_{\mathcal{A}}^{\text{sa}}(r).$$

(ii) *Any minimal realization*

$$\hat{\mathfrak{r}}(x) = \Delta + \hat{\Xi}^*(\hat{M}_0 - L_{\hat{M}}(x))^{-1}\hat{\Xi},$$

*of  $r$  satisfies the following property:*

*If  $\mathcal{A}$  is a unital complex  $*$ -algebra, which is stably finite, then*

$$(5.8) \quad \text{dom}_{\mathcal{A}}^{\text{sa}}(r) \subseteq \text{dom}_{\mathcal{A}}(\hat{\mathfrak{r}})$$

*and*

$$(5.9) \quad r(X) = \hat{\mathfrak{r}}(X) \quad \text{if } X \in \text{dom}_{\mathcal{A}}^{\text{sa}}(r).$$

*Proof.* For proving (i), we need a refinement of the argument that was used in the proof of Item (i) in Theorem 5.15: Since  $r$  is assumed to be regular at zero, we know that for any formal linear representation  $\rho_0 = (u_0, Q_0, v_0)$  of  $r - \Delta$ , the matrix  $Q_0^{(0)}$  appearing in the linear pencil

$$Q_0 = Q_0^{(0)} + Q_0^{(1)}x_1 + \dots + Q_0^{(g)}x_g$$

has to be invertible. Thus, we may form with  $\tilde{Q}_0^{(j)} := (Q_0^{(0)})^{-1}Q_0^{(j)}$  for  $j = 0, \dots, g$  the linear pencil

$$\tilde{Q}_0 = \tilde{Q}_0^{(0)} + \tilde{Q}_0^{(1)}x_1 + \dots + \tilde{Q}_0^{(g)}x_g \quad \text{where} \quad \tilde{Q}_0^{(0)} = I.$$

We define in addition  $\tilde{u}_0 := u_0$  and  $\tilde{v}_0 := (Q_0^{(0)})^{-1}v_0$ . Clearly, we obtain via this construction another formal linear representation  $\tilde{\rho}_0 = (\tilde{u}_0, \tilde{Q}_0, \tilde{v}_0)$  of  $r - \Delta$ . If we proceed now with the construction that was presented in (5.5), this yields a selfadjoint formal linear representation

$$\rho = (Q, v) := \left( \begin{pmatrix} 0 & \tilde{Q}_0^* \\ \tilde{Q}_0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2}\tilde{u}_0^* \\ \tilde{v}_0 \end{pmatrix} \right).$$

Now, we continue in analogy to the proof of Item (i) in Theorem 5.15. Starting with the selfadjoint formal linear representation  $\rho = (Q, v)$ , which can be seen also as a formal linear representation  $(v^*, Q, v)$ , we introduce

$$\mathfrak{r}_0(x) := -v^*(Q^{(0)} + Q^{(1)}x_1 + \cdots + Q^{(g)}x_g)^{-1}v,$$

which is of the form  $\mathfrak{r}_0(x) = \Xi^*(M_0 - L_M(x))^{-1}\Xi$  with  $\Xi = v$ ,  $M_0 = -Q^{(0)}$ , and  $M_j = Q^{(j)}$  for  $j = 1, \dots, n$ . Note that indeed  $M_0^2 = I$ . Finally, we put

$$\mathfrak{r}(x) := \Delta + \Xi^*(J - L_M(x))^{-1}\Xi.$$

Thus, by construction, we have for any unital complex  $*$ -algebra  $\mathcal{A}$  that

$$\text{dom}_{\mathcal{A}}^{\text{sa}}(r) \subseteq \text{dom}_{\mathcal{A}}(\mathfrak{r})$$

and

$$r(X) = \mathfrak{r}(X) \quad \text{if } X \in \text{dom}_{\mathcal{A}}^{\text{sa}}(r).$$

It remains to prove that  $\mathfrak{r}$  is indeed a realization of  $r$ . However, if applied in the case  $\mathcal{A} = M_N(\mathbb{C})$ , the statement above only yields that  $r$  and  $\mathfrak{r}$  take the same values on all selfadjoint matrices belonging to the domain of  $r$ , which does not allow to conclude directly that  $r$  and  $\mathfrak{r}$  are  $\mathbb{M}(\mathbb{R})$ -evaluation equivalent. In order to reach the desired conclusion, we need to take [HMV06, Proposition A.7] into account: Let  $\tilde{r}$  be any matrix-valued rational expression, which represents the noncommutative rational function that is determined by  $\mathfrak{r}$ . Thus, in other words,  $\mathfrak{r}$  is a descriptor realization of  $\tilde{r}$ . Since  $\tilde{r}$  and  $\mathfrak{r}$  are therefore  $\mathbb{M}(\mathbb{R})$ -evaluation equivalent and since  $r$  and  $\mathfrak{r}$  are  $\mathbb{M}(\mathbb{R})_{\text{sa}}$ -evaluation equivalent, we may conclude by transitivity that  $r$  and  $\tilde{r}$  are  $\mathbb{M}(\mathbb{R})_{\text{sa}}$ -evaluation equivalent. Now, [HMV06, Proposition A.7] tells us that  $r$  and  $\tilde{r}$  must be even more  $\mathbb{M}(\mathbb{R})$ -evaluation equivalent. Hence, again by transitivity, we obtain that  $r$  and  $\mathfrak{r}$  are in fact  $\mathbb{M}(\mathbb{R})$ -evaluation equivalent, as desired.

The assertion in (ii) can be proven as follows. If we assume that the selfadjoint rational expression  $r$  is regular at zero, we may consider besides its minimal selfadjoint realization

$$\hat{\mathfrak{r}}(x) = \Delta + \hat{\Xi}^*(\hat{J} - L_{\hat{M}}(x))^{-1}\hat{\Xi}$$

any other selfadjoint descriptor realization

$$\mathfrak{r}(x) = \Delta + \Xi^*(J - L_M(x))^{-1}\Xi,$$

with the prescribed feed through term  $\Delta$ , as constructed in (ii). Thus, if  $\mathcal{A}$  is any unital complex  $*$ -algebra, which is stably finite, we know

... by part (ii) that  $\text{dom}_{\mathcal{A}}^{\text{sa}}(r) \subseteq \text{dom}_{\mathcal{A}}(\mathfrak{r})$  holds and that we have

$$r(X) = \mathfrak{r}(X) \quad \text{for any } X \in \text{dom}_{\mathcal{A}}^{\text{sa}}(r).$$

... by Lemma 3.9 that  $\text{dom}_{\mathcal{A}}(\mathfrak{r}) \subseteq \text{dom}_{\mathcal{A}}(\hat{\mathfrak{r}})$  holds and that moreover

$$\mathfrak{r}(X) = \hat{\mathfrak{r}}(X) \quad \text{for any } X \in \text{dom}_{\mathcal{A}}^{\text{sa}}(r).$$

Combining both observation proves the stated inclusion (5.6) and also the representation given in (5.7).  $\square$

## 6. APPENDIX: PROOF OF THEOREM 2.3

For the readers convenience we now give in the regular case a proof of Theorem 2.3 (a special case of P.M. Cohn [Co06] Theorem 7.8.3) which we recall here.

### Theorem 6.1.

- (1) If  $\tilde{r}$  and  $r$  are  $\mathbb{M}(\mathbb{R})$ -evaluation equivalent, then  $\tilde{r}$  and  $r$  are  $\mathcal{A}$ -equivalent provided  $\mathcal{A}$  is a stably finite algebra.
- (2) If  $\mathcal{A}$  is not stably finite then there exists rational expressions  $\tilde{r}$  and  $r$ , which are  $\mathbb{M}(\mathbb{R})$ -evaluation equivalent, and  $X \in \mathcal{A}$  with  $X \in \text{dom}_{\mathcal{A}}(r) \cap \text{dom}_{\mathcal{A}}(\tilde{r})$ , but  $r(X) \neq \tilde{r}(X)$ .

*Proof.* 1) It suffices to check the following for any non-commutative rational function  $r$  which is  $\mathbb{M}(\mathbb{R})$ -evaluation equivalent to zero: if  $\mathcal{A}$  is a stably finite algebra and  $X = (X_1, \dots, X_g) \in \mathcal{A}$  is in the domain of  $r$ , then  $r(X) = 0$ . In order to prove this we realize  $r$  in the form  $r(x) = -uQ(x)^{-1}v$ . By Theorem 5.2 we know that we have such a realization where the domain of  $Q^{-1}$  contains the domain of  $r$ , thus we can assume that for our fixed  $X$  the matrix  $Q(X)$  is invertible in  $M_N(\mathcal{A})$ .

Since  $u, Q, v$  give a realization of  $r = 0$  its Sys matrix, as in equation (3.3),

$$\begin{pmatrix} Q(x) & v \\ u & 0 \end{pmatrix}$$

cannot be invertible. By results of Cohn [Co06] this means that it is not full, i.e. it can be written as the product of strictly rectangular matrices with non-commutative polynomials as entries. Since we are working with regular rational expressions, we can prove this directly from the machinery we have developed here, thus allowing the reader not to dig into the copious writings of Cohn.

First note that Remark 3.7 implies that  $\begin{pmatrix} Q(x) & v \\ u & 0 \end{pmatrix}$  can be written as a hollow matrix. Now hollow matrices factor as

$$\begin{pmatrix} f_{11} & f_{12} & f_{13} \\ 0 & 0 & f_{23} \\ 0 & 0 & f_{33} \end{pmatrix} = \begin{pmatrix} 1 & f_{13} \\ 0 & f_{23} \\ 0 & f_{23} \end{pmatrix} \begin{pmatrix} f_{11} & f_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This means we have

$$\begin{pmatrix} Q(x) & v \\ u & 0 \end{pmatrix} = \begin{pmatrix} P_1(x) & 0 \\ u_1(x) & 0 \end{pmatrix} \begin{pmatrix} P_2(x) & v_2(x) \\ 0 & 0 \end{pmatrix},$$

where  $P_1(x)$ ,  $P_2(x)$ ,  $u_1(x)$ ,  $v_2(x)$  are matrices with non-commutative polynomials as entries.

Replace  $x$  by  $X$  and apply the block LDU decomposition to get

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ 0 & -uQ(X)^{-1}v \end{pmatrix} &= \begin{pmatrix} Q(X)^{-1} & 0 \\ -uQ(X)^{-1} & 1 \end{pmatrix} \begin{pmatrix} Q(X) & v \\ u & 0 \end{pmatrix} \begin{pmatrix} 1 & -Q(X)^{-1}v \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} M(X) & 0 \\ m(X) & 0 \end{pmatrix} \begin{pmatrix} N(X) & n(X) \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Hence we have

$$\begin{pmatrix} 1 & 0 \\ 0 & r(X) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -uQ(X)^{-1}v \end{pmatrix} = \begin{pmatrix} M(X) & 0 \\ m(X) & 0 \end{pmatrix} \begin{pmatrix} N(X) & n(X) \\ 0 & 0 \end{pmatrix},$$

where  $M(X)$ ,  $m(X)$ ,  $N(X)$ ,  $n(X)$  are matrices over  $\mathcal{A}$ . This yields then

$$1 = M(X)N(X), \quad 0 = M(X)n(X), \quad 0 = m(X)N(X), \quad r(X) = m(X)n(X).$$

Since  $\mathcal{A}$  is stably finite the first equation implies that the square matrices  $M(X)$  and  $N(X)$  are inverses of each other, i.e., in particular, invertible; the second equation gives then  $n(X) = 0$ , which finally yields  $r(X) = 0$ .

2) If  $\mathcal{A}$  is not stably finite then there exists square  $n \times n$  matrices  $Q, P$  over  $\mathcal{A}$  such that  $PQ = 1$ , but  $QP \neq 1$ . Let  $T$  and  $S$  be  $n \times n$  matrices over the free field with indeterminate entries. We have then  $T(ST)^{-1}S - 1 = 0$ . This gives  $n^2$  equations in the entries of  $S$ ,  $T$  and  $(ST)^{-1}$ ; all of which are 0 in the free field. However, not all of them are true in our algebra  $\mathcal{A}$ , though all expressions make sense there.  $\square$

As the proof given above suggests, the statement (1) of Theorem 2.3 in fact also holds in the case where one compares rational expressions and their realizations under evaluation.

**Corollary 6.2.** *Let  $r$  be a matrix-valued rational expressions in formal variables  $x_1, \dots, x_g$  of dimension  $k \times k$ , which is regular at 0. Consider any realization  $\mathfrak{r}$  of  $r$ , i.e. any expression of the form*

$$\mathfrak{r}(x) = D + C(J - L_A(x))^{-1}B, \quad L_A(x) = A_1x_1 + \dots + A_gx_g,$$

*with  $N \times N$  matrices  $J$  and  $A_1, \dots, A_g$  where  $J$  satisfies  $J^2 = I_N$ , a  $N \times k$  matrix  $B$ , a  $k \times N$  matrix  $C$  and a  $k \times k$  matrix  $D$ , such that  $\mathfrak{r}$  and  $r$  are  $\mathbb{M}(\mathbb{R})$ -evaluation equivalent.*

*Provided that  $\mathcal{A}$  is a stably finite algebra, then  $r$  and  $\mathfrak{r}$  are also  $\mathcal{A}$ -evaluation equivalent, i.e., we have*

$$r(X) = \mathfrak{r}(X) \quad \text{for all } X \in \text{dom}_{\mathcal{A}}(r) \cap \text{dom}_{\mathcal{A}}(\mathfrak{r}),$$

*where*

$$\text{dom}_{\mathcal{A}}(\mathfrak{r}) := \{(X_1, \dots, X_g) \in \mathcal{A}^g \mid J - L_A(X) \in M_N(\mathcal{A}) \text{ is invertible}\}.$$

*Proof.* (i) For the given matrix-valued rational expression  $r$ , we may find according to part (i) of Theorem 5.15 a descriptor realization  $\tilde{\mathfrak{r}}$  of  $r$

$$\tilde{\mathfrak{r}}(x) = \tilde{D} + \tilde{C}(\tilde{J} - L_{\tilde{A}}(x))^{-1}\tilde{B}, \quad L_{\tilde{A}}(x) = \tilde{A}_1x_1 + \cdots + \tilde{A}_gx_g,$$

say of dimension  $\tilde{N}$ , which satisfies

$$\text{dom}_{\mathcal{A}}(r) \subseteq \text{dom}_{\mathcal{A}}(\tilde{\mathfrak{r}})$$

and

$$r(X) = \tilde{\mathfrak{r}}(X) \quad \text{for all } X \in \text{dom}_{\mathcal{A}}(r).$$

(ii) Now, since  $\mathfrak{r}$  and  $\tilde{\mathfrak{r}}$  are both realizations of the same rational expression  $r$ , we have that  $\mathfrak{r} - \tilde{\mathfrak{r}}$  is  $\mathbb{M}(\mathbb{R})$ -evaluation equivalent to 0. Thus,

$$(I_k \quad C \quad \tilde{C}) \left( \begin{pmatrix} I_k & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & \tilde{J} \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & L_A(x) & 0 \\ 0 & 0 & L_{\tilde{A}}(x) \end{pmatrix} \right)^{-1} \begin{pmatrix} D + \tilde{D} \\ B \\ -\tilde{B} \end{pmatrix}$$

yields a realization of 0, as it can be checked easily via  $\mathbb{M}(\mathbb{R})$ -evaluation equivalence. If we put  $u := (I_k \quad C \quad \tilde{C})$ ,

$$Q(x) := \begin{pmatrix} I_k & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & \tilde{J} \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & L_A(x) & 0 \\ 0 & 0 & L_{\tilde{A}}(x) \end{pmatrix}, \quad \text{and} \quad v := \begin{pmatrix} D + \tilde{D} \\ B \\ -\tilde{B} \end{pmatrix},$$

we may proceed as in the proof of Theorem 2.3: The fact that the matrix

$$\begin{pmatrix} Q(x) & v \\ u & 0 \end{pmatrix}$$

is hollow allows us to deduce according to Remark 3.7 that  $uQ(X)^{-1}v = 0$  holds, whenever  $X$  belongs to  $\text{dom}_{\mathcal{A}}(Q^{-1})$  for any stably finite algebra  $\mathcal{A}$ . Since

$$\begin{aligned} \text{dom}_{\mathcal{A}}(Q^{-1}) &= \text{dom}_{\mathcal{A}}((J - L_A(x))^{-1}) \cap \text{dom}_{\mathcal{A}}((\tilde{J} - L_{\tilde{A}}(x))^{-1}) \\ &= \text{dom}_{\mathcal{A}}(\mathfrak{r}) \cap \text{dom}_{\mathcal{A}}(\tilde{\mathfrak{r}}), \end{aligned}$$

we obtain

$$\tilde{\mathfrak{r}}(X) = \mathfrak{r}(X) \quad \text{for all } X \in \text{dom}_{\mathcal{A}}(\mathfrak{r}) \cap \text{dom}_{\mathcal{A}}(\tilde{\mathfrak{r}}).$$

(iii) Since  $\text{dom}_{\mathcal{A}}(r) \subseteq \text{dom}_{\mathcal{A}}(\tilde{\mathfrak{r}})$ , combining both observations that we made in (i) and (ii), leads us to

$$r(X) = \tilde{\mathfrak{r}}(X) = \mathfrak{r}(X) \quad \text{for all } X \in \text{dom}_{\mathcal{A}}(r) \cap \text{dom}_{\mathcal{A}}(\mathfrak{r}),$$

which finally proves the assertion.  $\square$

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